Mathematical Reviews

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Mathematical Reviews

Vol. 4, No. 10 — NOVEMBER, 1943 — Pages 265-292

NUMBER THEORY

Baur, Franz. Beiträge zum Problem der vollkommenen Zahlen. Deutsche Math. 6, 434–436, Berichtigung, 565 (1942). [MF 8616]

Known results (without reference) on odd perfect numbers. Addendum mentions Sylvester and Kanold.

L. Carlitz (Durham, N. C.).

Kantz, Georg. Über die Auflösung der Gleichung: $\phi(x)=n$, wenn $\phi(m)$ die Anzahl derjenigen natürlichen Zahlen bezeichnet, welche relativ prim zur natürlichen Zahl m und kleiner als m sind. Deutsche Math. 6, 437–449 (1942). [MF 8617]

Mainly known results, without any references. [For references, see Dickson, History of the Theory of Numbers, vol. 1, Stechert, New York, 1934, chap. 5.] *L. Carlitz.*

Mordell, L. J. Note on cubic Diophantine equations $z^2=f(x, y)$ with an infinity of integral solutions. J. London Math. Soc. 17, 199–203 (1942). [MF 8759]

The author notes that "many equations of the type $z^2=f(x, y)$, where $f(x, y)$ is a polynomial of the third degree in x, y, have an infinity of integer solutions" and supports this statement by geometrical considerations. He illustrates his methods by showing in detail that ~~the following equation has infinitely many integer solutions~~: certain equations of the form

$$z^2=ax^3+by^3+c,$$

have indefinitely many integer solutions, where each of a, b, c is a non-zero integer. The methods are essentially elementary. *B. W. Jones* (Ithaca, N. Y.).

Mordell, L. J. On Ryley's solution of $x^3+y^3+z^3=n$. J. London Math. Soc. 17, 194–196 (1942). [MF 8757]

The author quotes a result of S. Ryley, published in 1825, namely, that the equation

$$x^3+y^3+z^3=n, \qquad n \text{ rational},$$

has an infinity of rational solutions expressible as rational functions of a parameter. Using a device of Ryley, the author shows that the same statement can be made about the following equation:

$$(x+y+z)^3-dxyz=m,$$

where $d\neq 0$ and m are rational numbers. *B. W. Jones.*

Richmond, H. W. A note upon Prof. Mordell's paper. J. London Math. Soc. 17, 196–199 (1942). [MF 8758]

The author looks at Ryley's result [see the preceding review] from a geometrical point of view and refers to a fuller account of this argument and its results [Proc. Edinburgh Math. Soc. (2) 2, 92–100 (1930)].

B. W. Jones (Ithaca, N. Y.).

Ficken, Frederick A. Rosser's generalization of the Euclid algorithm. Duke Math. J. 10, 355–379 (1943). [MF 8474]

Rosser has given, in two papers [Proc. Nat. Acad. Sci. U. S. A. 27, 309–311 (1941); these Rev. 2, 349; Duke Math. J. 9, 59–95 (1942); these Rev. 3, 274], a generalization of the Euclid algorithm to sets of vectors in 2, 3, and 4 dimensional Euclidean space. The present paper gives a very complete discussion of this algorithm in the case in which it terminates (commensurable case). The algorithm in this case leads one from the given set of vectors through sets of shorter and shorter vectors to a terminal set which cannot be further shortened by the operations of the algorithm. Minimal properties of this terminal set lead to new proofs of the main theorems of Rosser, in some cases with improved hypotheses. *D. H. Lehmer* (Berkeley, Calif.).

Banerji, D. P. Congruence properties of Ramanujan's function $\tau(n)$. J. London Math. Soc. 17, 144–145 (1942). [MF 8260]

Ramanujan's function $\tau(n)$ is defined by

$$x[(1-x)(1-x^2)(1-x^3)\cdots]^{24}=\sum_1^\infty \tau(n)x^n.$$

The author proves two congruences which do not seem to have been noticed before, namely,

$$\tau(3m-r)\equiv 0 \pmod 3, \qquad r=0, 1,$$
$$\tau(4m-r)\equiv 0 \pmod 4, \qquad r=0, 1, 2.$$

If $r=0$, stronger results hold: $\tau(3m)\equiv 0 \pmod 9$ and $\tau(4m)\equiv 0 \pmod{64}$. *E. Hille* (New Haven, Conn.).

Titchmarsh, E. C. Some properties of the Riemann zeta-function. Quart. J. Math., Oxford Ser. 14, 16–26 (1943). [MF 8627]

The author considers the effect of the Riemann hypothesis on some asymptotic formulas concerning $\zeta(s)$, $M(x)=\sum_{n\leq x}\mu(n)$ and the zeros of $\zeta(s)$. He also considers the related hypothesis $\int_1^X(M(x)/x)^2dx=O(\log X)$, showing that it implies the Riemann hypothesis, that all the complex zeros of $\zeta(s)$ are simple and a number of other results. This hypothesis, which is less than $M(x)=O(x^{\frac{1}{2}})$, arises by analogy with a similar formula involving $\psi(x)-x$.

H. S. Zuckerman (Seattle, Wash.).

James, R. D. On the sieve method of Viggo Brun. Bull. Amer. Math. Soc. 49, 422–432 (1943). [MF 8391]

The object of the present paper is to show that the Eratosthenes sieve methods recently employed by Buchstab [C. R. (Doklady) Acad. Sci. URSS (N.S.) 29, 544–548 (1940); these Rev. 2, 348] and which apply to the natural series of primes may, with some modification, be made to apply to infinite subsets of the series of primes, such as primes in arithmetic progression. As a parallel to Buchstab's theorem that every sufficiently large even integer is the sum of two numbers having at most four prime factors, the author proves the following theorem: Let S be the set consisting of all products of at most 6 primes of the form $4n+1$, together with all numbers having two prime factors of the form $4n-1$ and at most three others of the form $4n+1$. Then every sufficiently large even number not divisible by 4 is the sum of two members of S.

D. H. Lehmer (Berkeley, Calif.).

Molsen, Karl. Ein Beitrag zur Irreduzibilität in algebraischen Zahlkörpern. Deutsche Math. 6, 449–452 (1942). [MF 8618]

Let $f(x)$ be a polynomial of the form

$$f(x)=1+a_1\frac{x}{1!}+a_2\frac{x^2}{2!}+\cdots+a_{n-1}\frac{x^{n-1}}{(n-1)!}\pm\frac{x^n}{n!}.$$

It was shown by I. Schur [S.-B. Preuss. Akad. Wiss. 1929, 125–136] that, if the a_i are rational integers, then $f(x)$ is irreducible in the field P of rational numbers. The author considers the case that the a_i are integers of an algebraic number field A. His main result is the following theorem: If B is an algebraic extension field of A, then there exists a number $\bar{n}(B)$ depending on B such that for $n\geqq\bar{n}(B)$ either the polynomial $f(x)$ is irreducible in B or $f(x)$ splits into an irreducible factor of degree $n-1$ and a linear factor both with coefficients in A. If $A=B$ is a quadratic field over P then there exists a number $\bar{n}(B)$ such that for $n\geqq\bar{n}(B)$ the polynomial $f(x)$ is irreducible in B. For the field $B=P(i)$, this number $\bar{n}(B)$ can be taken equal to 3. *R. Brauer.*

Rédei, L. Über den Euklidischen Algorithmus in reellquadratischen Zahlkörpern. J. Reine Angew. Math. 183, 183–192 (1941). [MF 8649]

This is essentially a translation into German of a paper in Hungarian [Mat. Fiz. Lapok 47, 78–90 (1940); these Rev. 2, 38]. Some of the proofs here are more elementary than before. *P. Erdös* (Philadelphia, Pa.).

Rédei, L. Zu einem Approximationssatz von Koksma. Math. Z. 48, 500–502 (1942). [MF 8724]

The purpose of this paper is to prove (1) that, if α is an algebraic integer whose conjugates have all moduli less than 1, α^n tends to zero (mod 1) as $n\to\infty$; (2) that the fractional part of $(p/q)^n$ $(p>q>1,\ p/q$ irreducible) has an infinite number of points of accumulation. Both results are known. [See Pisot, Ann. Scuola Norm. Super. Pisa (2) 7, 205–248 (1938); Vijayaraghavan, J. London Math. Soc. 15, 159–160 (1940); Proc. Cambridge Philos. Soc. 37, 349–357 (1941); these Rev. 2, 33; 3, 274.] *R. Salem.*

Pisot, C. Ein Kriterium für die algebraischen Zahlen. Math. Z. 48, 293–323 (1942). [MF 8556]

The main results of the paper are as follows. (I) Every algebraic number ξ of degree s can be approximated by a sequence of rational fractions u_n/v_n possessing the following properties: there exists a number $\alpha>1$, a positive ϵ and constants C_1, C_2, C_3, such that

$$(1)\qquad |u_{n+1}-\alpha u_n|<C_1/|u_n|^{\epsilon},\quad |v_{n+1}-\alpha v_n|<C_2/|v_n|^{\epsilon}$$

and

$$(2)\qquad |u_n-\xi v_n|<C_3/|v_n|^{\epsilon}.$$

Here $\epsilon\leqq(s-1)^{-1}$, but can be taken arbitrarily close to $(s-1)^{-1}$ by a suitable choice of α. (II) Every number ξ which can be approximated by a sequence of rational fractions u_n/v_n possessing the properties (1) is algebraic of degree $s\leqq(\epsilon+1)/\epsilon$, and the approximating fractions u_n/v_n possess the property (2). It is also shown that the converse proposition (II) can be improved by imposing on the approximating fractions conditions less restrictive than (1), that is, by supposing only

$$\sum_{k=n}^{2n}|u_{k+1}-\alpha u_k|^2\leqq 1/4\alpha^2,\quad \sum_{k=n}^{2n}|v_{k+1}-\alpha v_k|^2\leqq 1/4\alpha^2$$

for every n larger than a fixed m. The paper is closely related to a previous article of the author [Ann. Scoula Norm. Super. Pisa (2) 7, 205–248 (1938). *R. Salem.*

Pillai, S. S. Lattice points in a right-angled triangle. II. Proc. Indian Acad. Sci., Sect. A. 17, 58–61 (1943). [MF 8520]

Let ω and ω' be two positive numbers and denote by $L(\eta)$ the number of lattice points inside and on the boundary of the right triangle whose sides are the coordinate axes and the line $\omega x+\omega' y=\eta>0$. Write $F(\eta)=(\eta+\omega)(\eta+\omega')/2\omega\omega'$ and $\Delta(\eta)=L(\eta)-F(\eta)$. The author, using elementary methods, investigates the asymptotic behavior of $\Delta(\eta)$ as $\eta\to\infty$ obtaining results first proved (also with elementary methods) by Hardy and Littlewood [Abh. Math. Sem. Hansischen Univ. 1, 212–249 (1922)] and Ostrowski [Abh. Math. Sem. Hansischen Univ. 1, 77–98 (1922)]. The author's historical remark concerning his theorem IV is erroneous [compare theorem 10 in Hardy and Littlewood's paper cited above]. *D. C. Spencer* (Stanford University, Calif.).

Pillai, S. S. Lattice points in a right-angled triangle. III. Proc. Indian Acad. Sci., Sect. A. 17, 62–65 (1943). [MF 8521]

The author is here concerned with the error term $\Delta(\eta)$ when ω and ω' are co-prime positive integers [see the preceding review]. In this case G. H. Hardy [Ramanujan, Cambridge University Press, 1940, p. 73; these Rev. 3, 71] has shown that $\Delta(\eta)=-(\eta/\omega\omega')(\eta-[\eta]-\frac{1}{2})+E(\eta)$, where $E(\eta)=O(1)$ as $\eta\to\infty$. The author investigates the dependence of $E(\eta)$ on ω and ω', and proves that $E(\eta)$ is of the form $O(\omega+\omega')$. *D. C. Spencer* (Stanford University, Calif.).

THEORY OF GROUPS

Miller, G. A. Groups containing four and only four non-invariant subgroups. Proc. Nat. Acad. Sci. U.S.A. 29, 213–215 (1943). [MF 8715]

If a group G contains exactly four noninvariant subgroups, transformation of G by its own elements induces permutations on these four subgroups which form a permutation group P of degree 4. Groups G of the specified type are classified according to the nature of the group P as follows. (a) There are no groups G for which P is transitive. (b) There are three and only three groups G for which P is intransitive, of degree 4 and order 4. (c) There are two infinite sets of groups G for which P is intransitive, of degree 4 and order 2. The first of these systems contains only groups whose orders are a power of 2 while the second contains only groups whose orders are a power of 2 multiplied by an arbitrary odd prime. Finally the author points out that if a group contains exactly five noninvariant subgroups then these form a single set of five conjugates under transformation by the group. *D. C. Murdoch.*

Brauer, Richard. On permutation groups of prime degree and related classes of groups. Ann. of Math. (2) 44, 57–79 (1943). [MF 8075]

If a group G of order g contains elements P of prime order p which commute only with their own powers then $g=(p-1)p(1+np)/t$, where t and n are integers and $t\,|\,p-1$.

Transitive permutation groups of degree p, doubly transitive groups of degree $p+1$ and simple linear irreducible groups of degree p all have this property. Since $p|g$ to the first power only, the theory developed by the author in a former paper [Amer. J. Math. **64**, 401–440 (1942); these Rev. **4**, 1] is applicable and it is possible to limit the degrees of the irreducible representations of G which are prime to p. The main result of the paper is that, if G be further limited so that it coincide with its derived group, then for $n \geqq (p+3)/2$ we must have n expressible in the form $n \geqq$

$$\frac{puh+u^2+u+h}{u+1},$$

where u and h are positive integers. If $n<(p+3)/2$ there are two possibilities: (a) $n=1$, $t=2$ and $G=LF(2, p)$, where $p>3$; (b) $n=(p-3)/2$, $t=(p-1)/2$ and $G=LF(2, p-1)$, where $p=2^v+1$ a Fermat prime. *G. de B. Robinson.*

Venkatarayudu, T. Normal co-ordinates of symmetric point groups. II. Proc. Indian Acad. Sci., Sect. A. **17**, 75–78 (1943). [MF 8449]

Compare Young's semi-normal form of the irreducible representations of the symmetric group. The argument of this paper is no substitute. *G. de B. Robinson.*

Venkatarayudu, T. The character table of a subgroup of the symmetric group of degree 8. Proc. Indian Acad. Sci., Sect. A. **17**, 79–82 (1943). [MF 8450]

This subgroup is of order 64 and is generated by the permutations (15263748), (24)(57), (1234).

G. de B. Robinson (Ottawa, Ont.).

Hannink, Gunter. Verlagerung und Nichteinfachheit von Gruppen. Monatsh. Math. Phys. **50**, 207–233 (1942). [MF 8543]

Let U be a subgroup of finite index of a group G and let U' be the commutator group of U. Also let $\bar{g}$ be the representative of the left coset of U in G which contains the element g. Then the "Verlagerung" of G with respect to U is the mapping $x \rightarrow V(x)$ of G onto the Abelian group U/U', where $V(x)$ is defined by

$$V(x) = \prod_{\bar{g} (\mathrm{mod}\ U)} \bar{g} x \bar{g} \bar{x}^{-1} \quad \mathrm{mod}\ U'.$$

The author shows that many of the theorems on the existence of normal subgroups are easily proved by an appropriate Verlagerung. For example, he proves that every group of order $p_1^a p_2 \cdots p_r$ with $r \leqq p_1 < \cdots < p_r$ is solvable. Also he considers Burnside's conjecture that every simple group is of even order. *M. Hall* (Washington, D. C.).

Baer, Reinhold. Automorphism rings of primary Abelian operator groups. Ann. of Math. (2) **44**, 192–227 (1943). [MF 8283]

The purpose of the paper is an investigation of the automorphism rings of primary Abelian operator groups, leading to a characterization of such rings by peculiarities of their ideal theory. The author calls a ring E with an identity element a primary ring, if it contains a nilpotent two-sided ideal P such that every right ideal and every left ideal is a power of P. A primary Abelian operator group G is an Abelian group admitting the elements of such a ring E as operators. The rings to be studied are the rings $A=A(G, E)$ of all E-automorphisms of G. Of particular importance are special one-sided ideals of A, the left annulets and the right annulets. The left annulet $\Lambda(T)$ determined by a subset T of G is the set of all elements of A which carry G into T, while the right annulet $\mathrm{P}(T)$ consists of those elements of A which map T upon O. A left annulet can also be characterized as the set $\Lambda(\Sigma)$ of all left annihilators of a given subset Σ of A while a right annulet can be characterized as the set of $\mathrm{P}(\Sigma)$ of all right annihilators of Σ. If $D=D(G, E)$ is the modular lattice of all E-admissible subgroups S of G, then $S \rightarrow \Lambda(S)$ defines a projectivity of D upon the partially ordered set of all left annulets in A. On the other hand, $S \rightarrow \mathrm{P}(S)$ defines a duality of D upon the partially ordered set of all right annulets in A. It then is shown that the group of all projectivities of D is isomorphic to the group of all automorphisms of A, provided that D contains at least three independent cycles of order m, where m is the exponent of the radical P of E. Under the same assumptions concerning D, every automorphism of A is an inner automorphism of A if and only if every automorphism of the ring E is an inner automorphism. This theorem contains as a special case a known property of the automorphisms of normal simple algebras. Following these results, a large number of properties of the ideals, in particular of the annulets in A, is obtained. A set of seven of these properties is found to be a complete set of postulates for automorphism rings of primary Abelian operator groups. A ring K satisfies these postulates if and only if K is isomorphic to the ring $A(G, E)$ for a primary Abelian group G over a primary ring E, such that $D(G, E)$ contains at least two independent cycles of maximum order. These postulates are too complicated to be given here. It is shown that the structure of these rings is completely determined by their ideal theory.

R. Brauer (Toronto, Ont.).

Almeida Costa, A. On Abelian groups. Anais Fac. Ci. Pôrto **27**, 40 pp. (1942). (Portuguese) [MF 8012]

The subject of this pamphlet is a discussion of groups with operators; it follows, in general, the discussion in van der Waerden's Moderne Algebra [Springer, Berlin, 1930, sections 104 to 107]. The notations are different, and the presentation is more detailed than in the book. The author is particularly interested in the question of "invariance of dimensionality," that is, in situations when all bases have the same number of elements; he proves this invariance in the two cases when the ring of operators (assumed to have no divisors of zero) admits a division algorithm and has only principal ideals. In addition to references to van der Waerden's book there are references to the author's course "Elementos da Teoria dos Grupos" which was not accessible to the reviewer. *G. Y. Rainich.*

Malcev, A. On the structure of Lie groups in the large. C. R. (Doklady) Acad. Sci. URSS (N.S.) **37**, 3–5 (1942). [MF 8493]

The paper states (without proof) a number of interesting theorems on the topological structure of general Lie groups; in one or two cases the method of proof is very briefly indicated. Two theorems that indicate the algebraic properties of compact subgroups are: (1) all maximal compact connected subgroups of a Lie group are mutually conjugate; (2) if the factor group $\mathfrak{G}/\mathfrak{N}$ of the connected group $\mathfrak{G}$ by its connected closed invariant subgroup $\mathfrak{N}$ is compact, then there exists in $\mathfrak{G}$ a connected compact subgroup $\mathfrak{H}$ such that $\mathfrak{G}=\mathfrak{H}\mathfrak{N}$. The first of these theorems generalizes a result that was known to be valid for Abelian and semisimple groups. One theorem on the topological structure of the group space is (4) the topological space of any con-

nected Lie group is homeomorphic to the topological product of the space of its maximal compact subgroup and an Euclidean space. This was known to be valid for semi-simple groups and was extended to solvable groups by Chevalley [Ann. of Math. (2) **42**, 668–675 (1941); these Rev. **3**, 36].

The last part of the paper deals with closed subgroups of a group corresponding to a given Lie algebra G. The criteria are given in terms of characteristic subalgebras and rational submodulus in the commutative subalgebras A of G which the author defines. The final result is: the subgroup $\mathfrak{H}$ corresponding to a subalgebra H of G will be closed in every form of $\mathfrak{G}$ if and only if (1) $\mathfrak{H}$ is closed in the simply connected form $\bar{\mathfrak{G}}$ with algebra G, (2) $\mathfrak{H}$ is closed in the regular representation of G and (3) either H contains the center of G or $H\cap(A+\Gamma)=H\cap\Gamma$, for all characteristic subalgebras A, Γ being a certain maximal commutative subalgebra. *M. S. Knebelman* (Pullman, Wash.).

ANALYSIS

✶Margenau, Henry and Murphy, George Moseley. The Mathematics of Physics and Chemistry. D. Van Nostrand Co., Inc., New York, 1943. xii+581 pp. $6.50.

This is a very convenient collection of material from many fields of pure and applied mathematics, including thermodynamics, ordinary differential equations, vector analysis, curvilinear coordinates, calculus of variations, partial differential equations, eigenvalues and eigenfunctions, mechanics of molecules, matrices, quantum mechanics, statistical mechanics, numerical calculation, integral equations and group theory. The inclusion of all these useful topics, which are nowhere else to be found in a single volume, makes this a valuable reference work which should be owned by every advanced student of physics or chemistry. It may also be used as a text or a supplementary text in various courses, and most of the sections are provided with problems. It seems to be especially designed for use with a course in quantum mechanics.

The standard of rigor is that customary in technical books. This is not a handbook, however, and results are logically derived, not merely listed. Mathematicians will find the treatment of certain subjects, such as matrices and the calculus of variations, somewhat old-fashioned. The authors are rather vague on the question of which matrices may be reduced to diagonal form. The introduction of the concept of normal matrices would simplify the discussion, but the treatment of matrices is adequate for the applications intended. With some topics like quantum mechanics the physical theories involved are described in detail, while in other cases a mathematical theory is developed without much indication of where or how it is to be applied. A good feature is the unified treatment of the Legendre, Jacobi, Hermite and Laguerre polynomials and related functions, not readily available in other reference books. The authors' choice of topics is excellent. The only change the reviewer would make would be to include more complex variable theory. *O. Frink* (State College, Pa.).

Frucht, Roberto. On some inequalities. Math. Notae **3**, 41–46 (1943). (Spanish) [MF 8530]

Using a barycentric method, the author shows that, if $0<x\leqq x_i\leqq X$ and $0<m_i$ $(i=1, 2, \cdots, n)$, then

$$(1)\qquad 1\leqq\frac{(\sum m_ix_i)(\sum m_i/x_i)}{(\sum m_i)^2}\leqq\frac{(x+X)^2}{4xX},$$

and points out that the first part of (1) includes in particular two inequalities recently discussed by Saleme [Math. Notae **2**, 197–199 (1942)]. It might be remarked that the first part of (1) is the familiar inequality between the harmonic and arithmetic means, and therefore is a special case of Cauchy's inequality, and that the last part of (1) is included in a known inequality established by another method in Pólya and Szegö's "Aufgaben und Lehrsätze aus der Analysis" [Springer, Berlin, 1925, vol. I, p. 57]. The barycentric method previously has been used by Ş. A. Gheorghiu [Bull. Math. Soc. Roum. Sci. **35**, 117–119 (1933)] to obtain the sharpened form of Cauchy's inequality and also an analogously sharpened form of the Hölder-Jensen inequality. *E. F. Beckenbach* (Austin, Tex.).

Ritt, J. F. A family of functions and its theory of contact. Bull. Amer. Math. Soc. **49**, 109–113 (1943). [MF 7990]

On considère une expression de la forme

$$y(x)=\prod_{i=1}^{n}(x-a_i)^{p_i},$$

où les p_i sont des nombres positifs donnés et les a_i des constantes arbitraires; en supposant toutes les $a_i\neq 0$, $y(x)$ se développe au voisinage de $x=0$ en série entière $c_0+c_1x+\cdots+c_sx^s+\cdots$. On se propose de déterminer la valeur maxima de s telle que, pour des valeurs convenables des a_i, les coefficients $c_0, c_1, \cdots, c_s$ résultent tous moindres, en valeur absolue, qu'un nombre positif arbitrairement assigné. Pour $n=1$ il est évident que ce maximum est le plus grand entier positif moindre que $p=p_1$. L'auteur demontre que quel que soit n ce maximum est un entier $r\leqq q+n-1$, où q représente le plus grand entier moindre que $p_1+p_2+\cdots+p_n$, et que ce maximum est certainement atteint dans l'hypothèse que ne soit pas entière la somme d'une partie seulement des p_i; il n'est pas atteint si une puissance entière de $y(x)$ est un polynôme de degré moindre que $q+n-1$. La séparation des deux cas par une condition nécessaire et suffisante n'est pas obtenue. L'auteur donne au problème l'interprétation suivante: Étant donnée une famille de fonctions $g(x; a_1, a_2, \cdots)$ et une fonction fixe $f(x)$, appelons ordre de contact de la famille avec $f(x)$ en x_0 le plus grand nombre des dérivées successives de la différence $g-f$ que peuvent se reduire simultanément à un module aussi petit comme on veut, en x_0, par une élection convenable des paramètres a_i. Alors r est l'ordre de contact en $x=0$ de la famille $y(x)$ avec la fonction $y=0$. *B. Levi.*

Jung, F. Die Feldableitung. Deutsche Math. **6**, 524–530 (1942). [MF 8600]

The author gives an interesting geometric proof of the generalized Gauss theorem in three dimensional Euclidean space

$$\int_V dV\nabla * \Phi=\int_F df * \Phi.$$

Here Φ represents an arbitrary affinor; ∇, the del operator; $*$, a multiplication process (such as vector or scalar multiplication); the left hand side is a volume integral and the right hand side is a surface integral. In particular, the author gives an interpretation of the theorem for one dimension. It is remarked that by the same method of proof the theorem may be extended to n dimensions. *N. Coburn.*

Theory of Sets, Theory of Functions of Real Variables

Carathéodory, C. Gepaarte Mengen, Verbände, Somenringe. Math. Z. **48**, 4–26 (1942). [MF 8559]

A set M whose elements are paired by means of a symmetric relation is called a "paired set." If $A\subset M$ let $c(A)$ denote the set of all elements that are paired with every element of A. Then $A\subset c^2(A)$. The sets such that $A=c^2(A)$ are called "normal." The family of normal sets, ordered by inclusion, always forms a complete lattice L, and simple examples show that L is not in general a Boolean algebra. In any partially ordered set M, with least element o, a particular pairing is induced by the order relation, elements a and b being paired if they are "relatively prime," that is, such that $x\leqq a$ and $x\leqq b$ imply $x\leqq o$. The question arises, given an arbitrary paired set M, under what conditions is it possible to define a partial ordering in M such that the relatively prime pairs coincide with the given pairs. It is shown that this is possible when and only when the lattice L is a Boolean algebra, and then $c(A)$ coincides with the Boolean complement of A in L. When such an ordering is possible it is not unique, but the ordering defined by $a\leqq b$ if $c^2(a)\subset c^2(b)$ has the required properties and is the strongest such ordering. The relations between different possible orderings and the effect of identification of elements on the lattice L are discussed in detail. *J. C. Oxtoby.*

Erdös, P. and Tarski, A. On families of mutually exclusive sets. Ann. of Math. (2) **44**, 315–329 (1943). [MF 8287]

This paper deals with the problem of the existence, in a field of sets, of a family of mutually exclusive sets of maximum cardinal number. Given a field of sets F, let $m(F)$ stand for the least upper bound of the cardinal numbers of all families of mutually exclusive sets of F. The authors solve their problem by proving the following two results. (1) Except in the case where $m(F)$ is an inaccessible cardinal number greater than aleph-null, the field F always contains a family T of mutually exclusive sets of maximal cardinal number (that is, the cardinal number of T is $m(F)$). (2) If n is one of the hypothetical inaccessible cardinals greater than aleph-null, however, then there always exists a field F with $m(F)=n$, which does not contain any family T of mutually exclusive sets of maximal cardinal number $m(F)$. The authors state these results in a slightly different form, and they actually prove somewhat more; the statement in (1) is extended to rings of sets instead of fields, and, in fact, a similar result is proved for partially ordered sets with the proper definition of mutually exclusive elements. Likewise the field F whose existence is proved in (2) can be taken to be a complete Boolean algebra or can be replaced by the ring of open sets of a topological space. Finally the problem of the present paper is contrasted with other problems of general set theory which seem to involve the inaccessible cardinal numbers. *O. Frink* (State College, Pa.).

Rado, R. A theorem on independence relations. Quart. J. Math., Oxford Ser. **13**, 83–89 (1942). [MF 7632]

The notion of independence of elements $x_1, \cdots, x_m$ of an abstract set S is defined in terms of a function $I(x_1, \cdots, x_m)$, on the range of finite ordered subsets of S, having values 0, 1. An independence function I has the properties: (i) $I(x_1, \cdots, x_m)\geqq I(x_1, \cdots, x_m, x_{m+1})$; (ii) $I(x_1, \cdots, x_m)=I(x_{\nu_1}, \cdots, x_{\nu_m})$, $\nu_1, \cdots, \nu_m$ being a permutation of $1, \cdots, m$; (iii) $I(x, x)=0$;

$$\text{(iv)}\quad I(x_1, \cdots, x_m)I(y_1, \cdots, y_{m+1})\leqq \sum_{\mu=1}^{m+1} I(x_1, \cdots, x_m, y_\mu).$$

The relation between two properties H and R, defined in terms of a function I, are studied. The sets $A_1, \cdots, A_n$ have property H if, for $1\leqq k\leqq n$, $1\leqq\nu_1<\nu_2<\cdots<\nu_k\leqq n$, there is $(x_1, \cdots, x_k)\subset A_{\nu_1}+A_{\nu_2}+\cdots+A_{\nu_k}$ such that $I(x_1, \cdots, x_k)=1$. The sets A_ν have property R if there is $x_\nu\varepsilon A_\nu$, $\nu=1, \cdots, n$, such that $I(x_1, \cdots, x_n)=1$. It is shown that: (a) for every independence function I, H and R are equivalent; (b) if I satisfies (iii) and H and R are equivalent, then I is an independence function. An example of I satisfying (i), (ii), (iv) is given for which H and R are not equivalent. *L. W. Cohen* (Madison, Wis.).

Valentine, F. A. On the extension of a vector function so as to preserve a Lipschitz condition. Bull. Amer. Math. Soc. **49**, 100–108 (1943). [MF 7989]

Let x be a vector ranging over a two-dimensional Euclidean space V, and let $f(x)$ be a function defined over a set S in V and having vector values in V. The principal result in this paper is given in theorem 4. If $f(x)$ satisfies the Lipschitz condition

$$|f(x_1)-f(x_2)|\leqq K|x_1-x_2|$$

over S, then $f(x)$ can be extended over any set T in V containing S so as to preserve this Lipschitz condition, and this extension can be constructed so that the range of the extended function is contained in any given closed convex set containing the range of the original function. This theorem generalizes a similar result of McShane [Bull. Amer. Math. Soc. **40**, 837–842 (1934)] concerning real-valued functions over metric spaces. *C. C. Torrance.*

Tautz, Georg. Approximation von absolut additiven Mengenfunktionen durch absolut stetige. Deutsche Math. **6**, 553–558 (1942). [MF 8604]

The author considers an absolutely additive set function F of subsets ω of a square K and gives another proof of Radon's theorem that there exists a sequence F_n of absolutely continuous set functions such that $\int_K f(p)dF$ is the limit of $\int_K f(p)dF_n$ for every continuous point function f defined on K. To establish this result the author chooses a sequence ω_{0m} of B-measurable sets containing the origin and converging on it and then defines ω_{Pn} as the set obtained by translating ω_{0n} so that the origin moves to the point P. He then selects $F_n(\omega)$ to be

$$\int_\omega F(\omega_{Pn})/|\omega_{Pn}|d\omega_P.$$

It is shown that this sequence converges weakly to $F(\omega)$. *H. H. Goldstine* (Philadelphia, Pa.).

Helsel, R. G. and Radó, T. The transformation of double integrals. Trans. Amer. Math. Soc. **54**, 83–102 (1943). [MF 8712]

The authors consider the validity of the topological transformation formula

$$(1)\quad \iint_{S_0} F[x(u, v), y(u, v)]J(u, v; T)dudv = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(x, y)\mu(x, y; S_0, T)dxdy,$$

where T is a continuous transformation $x=x(u, v)$, $y=y(u, v)$

defined on the unit square S_0: $0\leqq u\leqq 1$, $0\leqq v\leqq 1$. The function F is measurable; J is the Jacobian $x_u y_v - x_v y_u$; $\mu(x, y; S_0, T)$ is the topological index of the point (x, y) with respect to the curve C, the image of the boundary of S_0 under T, if (x, y) is not on C and is zero otherwise. Radó and Reichelderfer [Trans. Amer. Math. Soc. **49**, 258–307 (1941); these Rev. **2**, 257] established the equality (1) under the assumption that the left side exists and the curve C is a set of planar measure zero for a wide class K_2 of transformations. In the present paper it is shown that the class K_2 is sufficiently broad to include all the results on the transformation of double integrals known to the authors. The results of McShane, Morrey, Rademacher, Radó, and Schauder follow easily from two results by Radó and Reichelderfer. To show that Young's transformations fall into class K_2 the authors approximate the given transformation by a sequence of ones in K_2 and satisfying a closure theorem which guarantees that the limit is again in K_2. The approximating transformations are formed by a highly skillful use of integral means, which form the basic tool of this paper. *H. H. Goldstine* (Philadelphia, Pa.).

Radó, Tibor. On a problem of Geöcze. Amer. J. Math. **65**, 361–381 (1943). [MF 8695]

Let S be a surface (in the sense of Fréchet) and let $A(S)$ denote its Lebesgue area. Let $A^*(S)$ denote the area obtained by requiring in the definition of Lebesgue area that each polyhedron be inscribed in S. The problem treated is the relation between $A(S)$ and $A^*(S)$. The result proved is more general than any heretofore proved in this connection and is as follows. Let S: $z=f(x, y)$, $(x, y)\in Q$: $0\leqq x\leqq 1$, $0\leqq y\leqq 1$, $f(x, y)$ continuous. Then $A^*(S)=A(S)$ if $f(x, y)$ is absolutely continuous (a.c.) in x for almost all y (or a.c. in y for almost all x). Obviously if $A(S)=+\infty$, $A^*(S)=+\infty$, since, in any case, $A^*(S)\geqq A(S)$. If S is of the form above and $A(S)<\infty$, then $f(x, y)$ is necessarily of bounded variation in the sense of Tonelli, which very nearly implies the hypothesis above.

The author indicates the construction of sequences of inscribed polyhedra whose projections are triangular subdivisions of the square Q formed by first dividing it into rectangles by lines parallel to the axes and then dividing each rectangle by the line joining the lower left and upper right vertices. For each rectangle r in Q, the writer defines two rectangle functions $n(r)$ and $\tau(r)=(\alpha^2+\beta^2\geqq\gamma^2)^{\frac{1}{2}}$, where $\alpha(r)$ is the area of the rectangle r: $x'\leqq x\leqq x''$, $y'\leqq y\leqq y''$, and

$$\beta(r)=\int_{y'}^{y''}[f(x'', y)-f(x', y)]dy,$$

$$\gamma(r)=\int_{x'}^{x''}[f(x, y'')-f(x, y')]dx.$$

For any sequence of subdivisions in which the maximum diameters of the r approach 0, it is known that $\sum\tau(r)\rightarrow A(S)$ [see Radó, Fund. Math. **10**, 197–210 (1927)]. The author constructs sequences of polyhedra of the type indicated above in which the differences between $\sum n(r)$ and the area of the corresponding polyhedron and also the differences $\sum n(r)-\sum\tau(r)$ tend to zero. It is shown further that these sequences may be chosen so that (1) the number of lines of each subdivision parallel to the x and y axes is the same for each subdivision, and (2) the ratio of the longest side of any of the rectangles of the jth subdivision to the shortest side of any of the rectangles of that subdivision approaches 1 as $j\rightarrow\infty$. *C. B. Morrey, Jr.*

Theory of Functions of Complex Variables

Badell, Enrique and González, Mario O. Computation of phase integral by a method of complex variables. Revista Soc. Cubana Ci. Fís. Mat. **1**, 37–41 (1942). (Spanish) [MF 8307]

The elementary integral

$$\int z^{-1}[(z-\alpha)(z-\beta)]^{-\frac{1}{2}}dz$$

is evaluated along the contour consisting of the straight line segment between the branch points α and β and return by the same line but on the other sheet of the Riemann surface. *P. W. Ketchum* (Urbana, Ill.).

Pisot, C. Über ganzwertige ganze Funktionen. Jber. Deutsch. Math. Verein. **52**, 95–102 (1942). [MF 8538]

Let $g(z)$ be an entire function having the property that $g(z)$ is an integer whenever z is a nonnegative integer. Let $M(r)$ be the maximum of $|g(z)|$ for $|z|=r$ and let

$$\overline{\lim_{r\rightarrow\infty}}\frac{\log M(r)}{r}\leqq\alpha.$$

Pólya proved in 1915 that, if $\alpha<\log 2$, $g(z)$ is a polynomial. Later Hardy, Pólya and Carlson proved that, if $\alpha=\log 2$, $g(z)$ is of the form $P_1(z)+2^zP_2(z)$, $P_1(z)$ and $P_2(z)$ being polynomials. Recently Selberg proved that this result is also true if only $\alpha\leqq\log 2+(1/1500)$ [see Arch. Math. Naturvid. **44**, 45–52 (1941); these Rev. **2**, 356, where Pólya suggests an alternative proof for Selberg's theorem].

The purpose of the present paper is to prove that, if $\alpha\leqq 0.8$, $g(z)$ is of the form

$$P_1(z)+2^zP_2(z)+\gamma^zP_3(z)+\bar{\gamma}^zP_4(z),$$

where the $P_k(z)$ are polynomials and γ, $\bar{\gamma}$ are the roots of $z^2-3z+3=0$. This result is itself a consequence of the following interesting theorem: there exists a constant α_0 $(0.825<\alpha_0<0.850)$ such that, if $\alpha\leqq\alpha_0$, $g(z)$ is of the form $\sum_1^3\beta_k^zP_k(z)$, the $P_k(z)$ being polynomials and the β_k certain algebraic numbers such that $|\log\beta_k|\leqq\alpha$. The idea of the proof is essentially the same as the idea of the alternative proof of Selberg's theorem sketched by Pólya [loc. cit. above]. *R. Salem* (Cambridge, Mass.).

Rademacher, Hans. On the Bloch-Landau constant. Amer. J. Math. **65**, 387–390 (1943). [MF 8697]

The upper bound $\mathfrak{L}<0.54326$ for the Bloch-Landau constant is proved. This is obtained by extending the results of Ahlfors and Grunsky [Math. Z. **42**, 671–673 (1937)] to obtain a suitable mapping of the unit circle on a Riemann surface. As is pointed out this mapping corresponds to that given by a certain modular function. The author mentions that this bound has also been found, but not published, by R. M. Robinson. *H. S. Zuckerman.*

Grunsky, Helmut. Eindeutige beschränkte Funktionen in mehrfach zusammenhängenden Gebieten. II. Jber. Deutsch. Math. Verein. **52**, 118–132 (1942). [MF 8540]

[The first part appeared in the same Jber. **50**, 230–255 (1940); these Rev. **2**, 275]. The following theorem which supplements part I of the present paper is established. Let B denote a plane region of connectivity $n+1$, the boundary of which consists of $n+1$ Jordan curves $C_0, C_1, \cdots, C_n$. Let

$\mathfrak{C}_\lambda$ ($\lambda=1, \cdots, l$) denote open disjoint subarcs of the C whose union together with their endpoints exhausts the boundary. Let m_λ denote given positive constants and finally let $z_0', z_0'', \cdots$ and z^* denote prescribed points of B which need not be distinct, though it is stipulated that only a finite number of them coincide; $K_{\rm IV}=K(\mathfrak{C}_\lambda, m_\lambda; z_0^{(\nu)})$ denotes the class of functions $F(z)$ which are regular and single-valued for z in B, have zeroes at the $z_0^{(\nu)}$ (the usual convention on multiple zeroes prevailing) and are such that

$$\overline{\lim_{z\to\mathfrak{C}_\lambda}}\, |F(z)|\leqq m_\lambda.$$

It is assumed that the distribution of the $z_0^{(\nu)}$ is such that $K_{\rm IV}$ contains functions not identically zero in B. Then, if j is the number of points $z_0^{(\nu)}$ which coincide with z^*, there exists at least one function $F_{\rm IV}(z)$ for which $|F^{(j)}(z^*)|$ attains an absolute maximum. On $\mathfrak{C}_\lambda$ the absolute value of every extremal function is m_λ (in the Fatou sense, if $\{z_0^{(\nu)}\}$ is not a finite set). In B every extremal function has at most n zeroes distinct from the $z_0^{(\nu)}$. For $n>0$ not all z^* have the same extremal functions.

The proof is based upon a study of a related problem in the theory of harmonic functions. The dependence of the periods of a conjugate function of a Green's function upon the position of the logarithmic singularity of the Green's function and the possibility of the approximation of non-positive harmonic functions in B by finite sums of simple types of harmonic functions are thoroughly utilized. The relation of the methods of the present paper to those used in inverting Abelian integrals is brought out. Allied questions are considered at the end of the paper.

M. H. Heins (Chicago, Ill.).

Schiffer, Menahem. The span of multiply connected domains. Duke Math. J. 10, 209–216 (1943). [MF 8461]

Let D_n be a multiply connected domain in the z-plane containing the point $z=\infty$; $\phi(D_n)$ is the family of functions $F(z)$ univalent and regular in D_n except at ∞ where $F(z)=z+A_2/z+\cdots$. It is known that there exist functions $f(z)=z+a_2/z+\cdots$ and $g(z)=z+b_2/z+\cdots$ belonging to $\phi(D_n)$ mapping D_n on regions bounded by horizontal and vertical slits, respectively, and maximizing and minimizing, respectively, $\Re(A_2)$. These are called the slit functions of D_n. The quantity $S(D_n)=\Re(a_2-b_2)$ is called the span of D_n.

Consider the class of regions D_n' conformally equivalent to D_n, that is, capable of being mapped on D_n by a function of $\phi(D_n)$. Let $F(D_n')$ be the inner measure of the complement CD_n' of D_n'. It follows from an inequality of Pólya that $F(D_n')$ is bounded from above. The author considers the properties of a region D_n for which $F(D_n)$ attains the maximum among all functions of the class. If z_0 is an interior point of CD_n and ρ is small, then $z^*=z+a\rho^2/(z-z_0)$, $|a|=1$, maps D_n on a conformally equivalent D_n^*. Using the fact that, for every ρ sufficiently small and all a such that $|a|=1$, $F(D_n^*)\leqq F(D_n)$, the author was able to prove that the slit functions for D_n satisfy

$$f'(z)+g'(z)=2,\quad f'(z)-g'(z)=-(2/\pi)\int_{CD_n} d\tau_x/(x-z)^2.$$

Comparing coefficients of $1/x^2$ in this last equation, the principal theorem of the paper results: $S(D_n)\geqq(2/\pi)F(D_n)$, or, the span of a domain is $2/\pi$ times the maximal area, the complement of which is conformally equivalent to the domain.

J. W. Green (Aberdeen, Md.).

v. Koppenfels, Werner. Bemerkungen zu der Arbeit von E. Graeser: Konforme Abbildung der längs eines beliebigen Kegelschnittbogens aufgeschlitzten Ebene auf das Äussere eines Kreises. Deutsche Math. 6, 558–564 (1942). [MF 8605]

The author points out that in certain symmetric special cases the integral involved in Graeser's formula for the mapping function [Deutsche Math. 2, 293–300 (1937)] becomes, under an elementary transformation, a Schwarz-Christoffel integral. This allows the parameters occurring in the formula to be calculated by known methods from the constants which characterize the region being mapped.

L. H. Loomis (Cambridge, Mass.).

Boas, R. P., Jr. Representation of functions by Lidstone series. Duke Math. J. 10, 239–245 (1943). [MF 8463]

The author gives necessary conditions and sufficient conditions for representation of a complex function $f(z)$ by a convergent (absolutely or conditionally) Lidstone series

$$(1)\quad f(z)=f(1)\Lambda_0(z)+f(0)\Lambda_0(1-z)+f''(1)\Lambda_1(z)+f''(0)\Lambda_1(1-z)+\cdots.$$

Here $\Lambda_n(z)$ is a polynomial of degree $2n+1$ such that $\Lambda_0(z)=z$, $\Lambda_n(0)=\Lambda_n(1)=0$, $\Lambda_n''(z)=\Lambda_{n-1}(z)$ $(n\geqq1)$. Illustration: (i) if the series (1) converges absolutely to $f(z)$, then $f(z)=O(e^{\pi|z|})$, $|z|\to\infty$; (ii) the series (1) converges to $f(z)$ if $f(z)=O(z^{-\frac{1}{2}}e^{\pi|z|})$, $|z|\to\infty$. The discussion is based upon some results of Widder concerning Lidstone series [Trans. Amer. Math. Soc. 51, 387–398 (1942); these Rev. 3, 293], particularly upon the trigonometric series representation of $\Lambda_n(iy)$.

J. A. Shohat (Philadelphia, Pa.).

Kober, H. On the approximation to integrable functions by integral functions. Trans. Amer. Math. Soc. 54, 70–82 (1943). [MF 8711]

The author considers the problem of approximating functions on a line or half-line by entire functions. The possibility of approximating a continuous function arbitrarily closely by entire functions was shown by Carleman [Ark. Mat. Astr. Fys. 20B, no. 4, 1–5 (1927); see also A. Roth, Comment. Math. Helv. 11, 77–125 (1938)], but the author's results not only settle the problem for L^p approximation but give the smallest possible order for the approximating functions. His principal result is that a function of L^p $(0<p<\infty)$ can be approximated in the L^p metric by entire functions of order 1 or $\frac{1}{2}$, respectively, and finite (but unbounded) type; if it can be approximated in the same sense by functions of smaller order, it is almost everywhere zero. A bounded function $f(t)$ can be uniformly approximated by functions of order 1 or $\frac{1}{2}$ if and only $f(t)$ or $f(t^2)$ (respectively) is uniformly continuous. Functions of best approximation are also discussed. A preliminary part of the paper is devoted to a study of the spaces whose elements are entire functions of order 1 or $\frac{1}{2}$ and finite type, belonging to L^p in $(-\infty,\infty)$ or $(0,\infty)$. An essential part is played by approximating functions (on $(-\infty,\infty)$) of the form $\alpha\int_{-\infty}^{\infty}\kappa\{\alpha(t-x)\}f(t)dt$, κ entire, of order 1.

R. P. Boas, Jr.

Fourier Series and Generalizations, Integral Transforms

Erdös, P. On the convergence of trigonometric series. J. Math. Phys. Mass. Inst. Tech. 22, 37–39 (1943). [MF 8769]

Let $\{n_k\}$ be a sequence of positive integers such that the number of solutions of $n_i+n_j=a$ is uniformly bounded for

all integers a. Then the series $\sum \rho_k \cos (n_k x - \alpha_k)$ converges almost everywhere if $\sum \rho_k^2 < \infty$. In other words, the series behaves like a lacunary series. It is to be observed that there are sequences of the above described type for which $n_k = O(k^2)$. The method of the proof is new and interesting in itself. *R. Salem* (Cambridge, Mass.).

Wang, Fu Traing. A note on Cesàro summability of Fourier series. Ann. of Math. (2) **44**, 397–400 (1943). [MF 8869]

Let

$$(*)\qquad \varphi(t) \sim \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nt,$$

and suppose that

$$\int_0^t (t-u)^{\beta-1}\{\varphi(u) - s\}du = o(t^\gamma), \qquad \gamma > \beta > 0.$$

Let n be a positive integer such that $n \geqq \beta > n-1$. Then the series (*) is summable (C, δ) to s at $t=0$, where $\delta = (\gamma(n-1)+\beta)/(\gamma+n-\beta)$. *A. Zygmund.*

Wang, Fu Traing. Some results on Riesz's summability of Fourier series. Acad. Sinica Science Record **1**, 42–44 (1942). [MF 8832]

Preliminary report on results which the author has meanwhile published elsewhere [Proc. London Math. Soc. (2) **47**, 308–325 (1942); J. London Math. Soc. **17**, 98–107 (1942); these Rev. **4**, 37]. *A. Zygmund.*

Cheng, Min-Teh. On strong summability of Fourier series. Acad. Sinica Science Record **1**, 91–97 (1942). [MF 8845]

Hardy and Littlewood have proved [Fund. Math. **25**, 162–189 (1935)] that the partial sums $S_\nu(x)$ of the Fourier series of a function f satisfy the condition

$$\sum_{\nu=0}^{n} |S_\nu(x) - f(x)|^k = o(n(\log n)^{k/2}), \qquad k \geqq 2,$$

at every point x where

$$(*)\quad \int_0^t |\varphi_x(u)|du = o(t), \quad \varphi_x(u) = f(x+u) + f(x-u) - 2f(x).$$

For k even, the author replaces the condition (*) by

$$\int_0^t \varphi_x(u)du = o(t), \quad \int_0^t |\varphi_x(u)|du = O(t).$$

A. Zygmund (South Hadley, Mass.).

Chen, Kien-Kwong. On the convergence of the conjugate series of a Fourier series. Acad. Sinica Science Record **1**, 1–6 (1942). [MF 8823]

A slight extension of the analogue for conjugate series of the very well-known Lebesgue test. *A. Zygmund.*

Hedge, L. B. Transformations of multiple Fourier series. Bull. Amer. Math. Soc. **49**, 262–269 (1943). [MF 8222]

The author generalizes, from one to several variables, known theorems on sequences of factors which transform Fourier series of one class into those of another class. For instance, the transformations (B_1B), (C_1C), (L_1L), (S_1S) are again all alike, the factors being the Fourier coefficients of a Fourier-Stieltjes series. Actually, the results as stated by the author are valid for Fourier series on any compact group, not necessarily torus spaces. *S. Bochner.*

Bellman, Richard. Fourier integrals. Duke Math. J. **10**, 247–248 (1943). [MF 8464]

By analogy with the case of Fourier series the author establishes the following gap theorem for the Fourier integral. Let $f(x)$ belong to $L^p(0, \infty)$, $1 < p \leqq 2$, and $\{n_k\}$ be a sequence satisfying $n_{k+1}/n_k \geqq \lambda > 1$; if

$$s_n = (2\pi)^{-\frac{1}{2}} \int_0^n f(x)e^{ixt}dx,$$

$F(t) = \text{l.i.m. } s_n(t)$, then $\lim_{k\to\infty} s_{n_k}(t) = F(t)$ almost everywhere. The proof depends on an integral analogue of the Young-Hausdorff theorem due to Titchmarsh [Proc. London Math. Soc. (2) **23**, 279–289 (1924)].

[It is of interest to compare this with a nongap result of Zygmund [Proc. Cambridge Philos. Soc. **32**, 321–327 (1936)]. If $f(x)$ belongs to L^p, $1<p<2$, then

$$\lim_{\lambda\to\infty} (2\pi)^{-\frac{1}{2}} \int_{-\lambda}^{\lambda} f(x)e^{ixt}dx = F(t)$$

almost everywhere, where $F(t)$ is the Fourier transform of $f(x)$.] *H. Pollard* (Gambier, Ohio).

Fuchs, W. H. J. and Rogosinski, W. W. A note on Mercer's theorem. J. London Math. Soc. **17**, 204–210 (1942). [MF 8760]

Consider the transform

$$t(x) = s(x) - ((\zeta+1)/x)\int_0^x s(t)dt,$$

where ζ is a complex number. Mercer's theorem is that, if $s(x) \to s$ $(x \to \infty)$, then $t(x) \to -\zeta s$, and that the converse holds if and only if $\Re(\zeta) < 0$. The transform thus defines a Hausdorff summation method M_ζ which is stronger than convergence if and only if $\Re(\zeta) \geqq 0$. A result of Hardy determines the functions $s(x)$ which are summable M_ζ if $\Re(\zeta) > 0$, namely, those of the form $A_0 x^\zeta + c(x)$, where $c(x)$ approaches a limit. The authors' main results are characterizations of the functions which are summable by the iterates M_ζ^p of the method M_ζ. If $\Re(\zeta) > 0$, the form is

$$s(x) = x^\zeta \sum_{r=0}^{p-1} A_r \log^r x + c(x),$$

where $c(x)$ approaches a limit. If $\Re(\zeta) = 0$, the form of $s(x)$ is more complicated. The results are applied to the study of continuous Hausdorff methods T which can be factored into the form $M_\zeta^p T_1$, these being the methods whose associated Mellin transforms have a zero of order p at ζ. *R. P. Boas, Jr.* (Cambridge, Mass.).

Bary, Nina. Sur la stabilité de la propriété d'être un système complet de fonctions. C. R. (Doklady) Acad. Sci. URSS (N.S.) **37**, 83–87 (1942). [MF 8497]

Let $\{x_n\}$, $\{y_n\}$ be sequences of elements in Hilbert space. Define $d = \sum_1^\infty \|x_n - y_n\|^2$. Then the paper studies the relation between the completeness (in the sense of orthogonal functions) of $\{x_n\}$ and $\{y_n\}$ under the hypothesis $d < \infty$. Here $\{x_n\}$ is said to be a linearly independent set if $\sum_1^\infty a_n x_n = 0$ implies all the a_n are zero; $\{x_n\}$ is defined to be minimal if no x_n can be approximated with arbitrary accuracy by finite linear combinations of the others.

The following results are obtained. (i) If $\{x_n\}$, $\{y_n\}$ are orthonormal and $d < \infty$, then both systems are complete or both incomplete. This is a best possible result. (ii) If $\{x_n\}$ is a complete orthonormal set and $\{y_n\}$ is normal,

$d<\infty$, then $\{y_n\}$ is complete if and only if $\{y_n\}$ is a linearly independent set. (iii) If $\{x_n\}$ is complete and $\{y_n\}$ is any set such that $d<\infty$, then $\{y_n\}$ is complete if and only if it is minimal. The proof depends on results of von Koch on infinite systems of linear equations. It is to be noted that in the double sums $\sum_k\sum_n$ which occur k should be different from n. *H. Pollard* (Gambier, Ohio).

Polynomials, Polynomial Approximations

Maximoff, I. On neighbouring roots. C. R. (Doklady) Acad. Sci. URSS (N.S.) **37**, 88–90 (1942). [MF 8498]

A system $x_1, x_2, \cdots, x_k$ of real roots of a real equation $f(x)=0$ is said to form a complete set of neighboring roots of order k if $x_k-x_{k-1}=x_{k-1}-x_{k-2}=\cdots=x_2-x_1=h>0$, but $f(x_1-h)f(x_k+h)\neq 0$. In this paper it is proved that, if $x_1, x_2, \cdots, x_k$ is a complete set of neighboring roots of the equation $f(x)=0$, then $x_1, x_2, \cdots, x_{k-1}$ is a complete set of neighboring roots for the equation $\Delta f(x)\equiv f(x+h)-f(x)=0$. Thus $f(x)$ may be factored according to its complete sets of neighboring roots by repeated formation of the quotient $f_1(x)=f(x)/D(x)$, where $D(x)$ is the greatest common divisor of $f(x)$ and $\Delta f(x)$. Finally, introducing the differences $\Delta^{(p)}f(x)=\Delta^{(p-1)}f(x+h)-\Delta^{(p-1)}f(x)$ with $\Delta^0 f(x)=f(x)$, the author proves that for $x_1, x_2, \cdots, x_k$ to form a complete set of neighboring roots of $f(x)=0$ it is necessary and sufficient that $\Delta^{(p)}f(x_1)=0$ for $p=0, 1, 2, \cdots, k-1$ with $f(x_1-h)\Delta^{(k)}f(x_1)\neq 0$. This theorem may be applied in order to find, for a given $h>0$, the complete set of roots to which a given root of $f(x)=0$ belongs. *M. Marden.*

Kneser, Hellmuth. Zur Stetigkeit der Wurzeln einer algebraischen Gleichung. Math. Z. **48**, 101–104 (1942). [MF 8563]

The usual theorem on the continuity of the roots of an algebraic equation provides that, if the polynomial

$$f(x)=a_0+a_1x+\cdots+a_{n-1}x^{n-1}+x^n$$

has a zero at $x=c$, then

$$g(x)=b_0+b_1x+\cdots+b_{n-1}x^{n-1}+x^n$$

will have a zero arbitrarily close to $x=c$, provided that all the differences $|b_k-a_k|$ are taken sufficiently small. In this paper the following stronger theorem is given. If $f(x)$ has an m-fold zero at $x=c$, then $g(x)$ has at least m zeros in an arbitrarily small neighborhood of $x=c$ provided that the differences $|b_k-a_k|$ merely for $k=0, 1, \cdots, m-1$ are taken sufficiently small. For the case $m=1$, this theorem is implied in the continuity proof in van der Waerden's "Einführung in die algebraische Geometrie" [Berlin, 1939, p. 48]. The general case is established by mathematical induction. A consequence of the theorem is the following. The polynomial $f(x)$ has at least m zeros in an arbitrarily small neighborhood of a point $x=c$ if, at $x=c$, $f(x)$ and its first $m-1$ derivatives all assume sufficiently small values. *M. Marden* (Milwaukee, Wis.).

Szász, Otto. On sequences of polynomials and the distribution of their zeros. Bull. Amer. Math. Soc. **49**, 377–383 (1943). [MF 8381]

The author gives a proof for a theorem of Lindwart-Pólya under a weaker hypothesis. The underlying result is as follows. Let $\{P_n(z)\}$ be a sequence of polynomials of the precise degree $m_n\uparrow\infty$, with roots in the half-plane $\Re(e^{i\theta_n}z)\geqq 0$, $z\neq 0$. Furthermore, let the quantities $P_n(0)$, $\{P_n(0)\}^{-1}$, $P_n'(0)$, $P_n''(0)$ be bounded. Then the given sequence of polynomials is uniformly bounded in any circle $|z|\leqq r$. *G. Szegö* (Stanford University, Calif.).

Dickinson, D. R. On Tchebycheff polynomials. III. J. London Math. Soc. **17**, 211–217 (1942). [MF 8761]

[The first two parts appeared in the Quart. J. Math., Oxford Ser. **10**, 277–282 (1939); **12**, 184–192 (1941); these Rev. **1**, 143; **3**, 236.] The present paper deals with points on a given interval $[a, b]$ which can serve as strong multiple zeros of a given T-system of polynomials on $[a, b]$. [For definitions and notations cf. parts I and II, particularly II.] It is shown that the set of such points is either finite or denumerably infinite. The proof is based upon a detailed discussion of sets $\{W_s\}$, $s=1, 2, \cdots, n$, consisting of points $\bar{x}$ in $[a, b]$ such that there exists a member in our given system having at $\bar{x}$ a weak zero of order r. *J. Shohat.*

Erdös, P. On some convergence properties of the interpolation polynomials. Ann. of Math. (2) **44**, 330–337 (1943). [MF 8288]

Let $l_i^{(n)}(x)$, $i=1, 2, \cdots, n$, be the fundamental polynomials of degree $n-1$ associated with the points of interpolation $x_1^{(1)}$; $x_1^{(2)}, x_2^{(2)}$; $\cdots$ in the interval $(-1, 1)$. It is shown that if the fundamental functions $l_i^{(n)}(x)$ are uniformly bounded in $(-1, 1)$ then for every continuous function $f(x)$ and for every $c>0$ there is a sequence of polynomials $\phi_n(x)$ which approximate $f(x)$ in the following sense: (1) the degree of $\phi_n(x)$ is equal to or less than $n(1+c)$; (2) $\phi_n(x)$ coincides with $f(x)$ at the fundamental points $x_1^{(n)}, x_2^{(n)}, \cdots, x_n^{(n)}$; (3) $\phi_n(x)$ converges uniformly to $f(x)$ in $(-1, 1)$. To prove this result the function $f(x)$ is first approximated within a distance ϵ by some polynomial $\psi_{n-1}(x)$ of degree $n-1$. If ϵ_i is equal to the difference $f(x)-\psi_{n-1}(x)$ at the point $x_i^{(n)}$ then it is necessary to add to $\psi_{n-1}(x)$ a polynomial of the form $\sum\epsilon_i l_i^{(n)}(x)h_i(x)$, where $h_i(x)$ is a polynomial equal to 1 at the point $x_i^{(n)}$ and "small" outside a neighborhood of this point. If the fundamental points $x_i^{(n)}$ are equally spaced in the interval $(-1, 1)$ it is shown that an arbitrary continuous function $f(x)$ may be uniformly approximated in the interval by a sequence of polynomials of degree $\pi n(1+c)/2$ which coincide with $f(x)$ at the fundamental points. It is further shown that no lower constant than $\pi n/2$ suffices. This answers a question which was raised by Fejér. *A. C. Schaeffer.*

Kowalewski, Gerhard. Über das neue Theorem von Obreschkoff. Deutsche Math. **6**, 349–351 (1942). [MF 8610]

A new simple derivation of a formula of Obreschkoff's generalization of Taylor's formula [Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. **1940**, no. 4 (1940); these Rev. **2**, 284]. *J. A. Shohat* (Philadelphia, Pa.).

Schmeidler, Werner. Über ein zweidimensionales Analogon einer Formel der Integralrechnung. J. Reine Angew. Math. **183**, 175–182 (1941). [MF 8648]

The integral formula

$$\int_0^\pi \frac{\cos n\varphi}{\cos\varphi-\cos\psi}d\varphi=\frac{\pi\sin n\varphi}{\sin\varphi},\quad n=0, 1, 2, \cdots,$$

can, by a proper change of variables, be put in the form

$$-\frac{1}{\pi}\int_{-1}^{+1}\frac{d}{dx}((1-x^2)^{\frac{1}{2}}P_n(x))\frac{dx}{x-\xi}=(n+1)P_n(\xi),$$

where $\{P_n\}$ is an orthogonal set of Gegenbauer polynomials. In the present paper the author considers a two dimensional analogue of the above integral

$$I=-\frac{1}{4\pi}\int\int\left\{\frac{\partial}{\partial x}\left(\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{\frac{1}{2}}P_{m,n}\left(\frac{x}{a},\frac{y}{b}\right)\right)\frac{x-\xi}{r^3}\right.$$
$$\left.+\frac{\partial}{\partial y}\left(\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{\frac{1}{2}}P_{m,n}\left(\frac{x}{a},\frac{y}{b}\right)\right)\frac{y-\eta}{r^3}\right\}dxdy,$$

where the integration is performed over the ellipse $x^2/a^2+y^2/b^2=1$ and $P_{n,m}$ is a member of a biorthogonal system obtained by Koschmieder as an extension of the Gegenbauer polynomial. It is found that the integral I is equal to a polynomial in ξ/a, η/b of degree $n+m$, whose coefficients are expressible in terms of complete elliptic integrals of the first and second kind.

An application is made of the results to the solution in terms of a series of biorthogonal polynomials of the problem of three dimensional flow about an elliptical plate placed edgewise in the flow and slightly distorted. *J. W. Green.*

Special Functions

Hille, E. and Szegö, G. On the complex zeros of the Bessel functions. Bull. Amer. Math. Soc. **49**, 605–610 (1943). [MF 8862]

Let $t=z^{\frac{1}{2}}$; then if z is bounded and if α is an arbitrary real number,

$$n^{-\alpha}L_n^{(\alpha)}(z/n)\to t^{-\alpha}J_\alpha(2t) \qquad \text{uniformly as } n\to\infty.$$

This theorem, by which the Bessel function is represented as a limiting form of a generalised polynomial of Laguerre, is used here to obtain a new proof of the theorem that when $\beta\geqq 0$ the entire function $t^\beta J_{-\beta}(2t)$ has precisely $[\beta]$ nonpositive zeros. Use is made of a lemma which states that when $n\geqq 1$ and $\alpha\neq -1, -2, -n$ the zeros of $L_n^{(\alpha)}(x)$ are distinct and not equal to zero. The number of positive zeros is n if $\alpha>-1$; it is $n+[\alpha]+1$ if $-n<\alpha<-1$; it is 0 if $\alpha<-n$. The number of negative zeros is 0 or 1. Bounds are also given for the nonpositive zeros of $L_n^{(\alpha)}(x)$. In the analysis much use is made of the differential equation $w''+G(x)w=0$ which is satisfied by the function

$$w(x)=e^{-x/2}x^{(\alpha+1)/2}L_n^{(\alpha)}(x).$$

One deduction from the differential equation is that when integration is extended over an arbitrary rectifiable curve in the complex x-plane from A to B

$$[\bar{w}w']_A^B-\int|w'|^2d\bar{x}+\int G(x)|w|^2dx=0.$$

H. Bateman (Pasadena, Calif.).

Truell, Rohn. Concerning the roots of $J_n'(x)N_n'(kx)-J_n'(kx)N_n'(x)=0$. J. Appl. Phys. **14**, 350–352 (1943).

An asymptotic formula for the zeros of the equation in the title was given by McMahon [Ann. of Math. (2) **9**, 23–25 (1895)]. The author shows that this formula does not include the smallest zeros except when $n=0$. By a graphical method he obtained curves showing the roots of the equation as a function of $k(>1)$ for $n=1, 2, 3, 4$. He also proves that, when $n\neq 0$, the function $J_n'(x)/N_n'(x)$ has one and only one relative maximum at $x=n$. *M. C. Gray.*

Brauer, P. und E. Berichtigungen zu: Über unvollständige Anger-Webersche Funktionen. Z. Angew. Math. Mech. **22**, 304 (1942). [MF 8916]

This paper appeared in the same Z. **21**, 177–182 (1941); these Rev. **3**, 116.

Straubel, Rudolf. Unbestimmte Integrale mit Produkten von Zylinderfunktionen. II. Ing.-Arch. **13**, 14–20 (1942). [MF 8595]

The starting point of the paper is an identity between indefinite integrals of: (1) products of two cylindrical functions of different orders and proportional arguments, (2) products of their derivatives, (3) products of one cylindrical function by the derivative of the other. This identity involves four arbitrary functions. Particular formulas which require a suitable choice of these functions are derived.

I. Opatowski (Chicago, Ill.).

Jackson, Dunham. Legendre functions of the second kind and related functions. Amer. Math. Monthly **50**, 291–302 (1943). [MF 8331]

An introductory treatment of the general solution of Legendre's differential equation. A similar treatment is indicated for the differential equation of the associated functions and for the Jacobi, Hermite and Laguerre equations and finally also for Bessel's differential equation.

G. Szegö (Stanford University, Calif.).

MacRobert, T. M. Associated Legendre functions of the first kind when the sum of the degree and the order is a positive integer. Quart. J. Math., Oxford Ser. **14**, 1–4 (1943). [MF 8625]

The author obtains polynomial expansions for the associated Legendre functions $P_{m+n}^{-m}(\mu)$ (m arbitrary, n an integer) and for the related functions $T_{n+m}^{-m}(\mu)$, where

$$T_n^{-m}(\mu)=\frac{1}{\Gamma(m+1)}\left(\frac{1-\mu}{1+\mu}\right)^{m/2}F(-n, n+1; m+1; \tfrac{1}{2}-\tfrac{1}{2}\mu).$$

In particular, the relations

$$T_{m+2p}^{-m}(0)=(-1)^p\frac{(2p)!}{2^{m+2p}\Gamma(m+p+1)p!}, \quad T_{m+2p+1}^{-m}(0)=0$$

are used to obtain the expansion

$$T_{m+n}^{-m}(\cos\theta)=\frac{\sin^m\theta\cos^n\theta}{2^m\Gamma(m+1)}F(-\tfrac{1}{2}n, \tfrac{1}{2}-\tfrac{1}{2}n; m+1; -\tan^2\theta).$$

M. C. Gray (New York, N. Y.).

MacRobert, T. M. Proofs of some formulae for the hypergeometric function and the E-function. Philos. Mag. (7) **34**, 422–426 (1943). [MF 8738]

The author first provides alternative proofs for some known theorems on hypergeometric functions, then gives a formula for an integral involving the product of two E-functions. From this a formula for the integral

$$\int_0^\infty t^{l-1}K_m(xt)K_n(t)dt$$

given by Titchmarsh [J. London Math. Soc. **2**, 98 (1926)] may be obtained as a special case. *M. C. Gray.*

Dhar, S. C. Integral representations of Whittaker and Weber functions. J. Indian Math. Soc. (N.S.) 6, 181–185 (1942). [MF 8377]

A typical integral is

$$W_{\alpha,\beta}(z)=\frac{z^{\alpha-\gamma}}{\Gamma(2\gamma-2\alpha)}\int_0^\infty e^{-\frac{1}{2}s}s^{2\gamma-\alpha-1}\left(1+\frac{s}{z}\right)^{-\frac{1}{2}-\delta}W_{\gamma,\delta}(z+s)ds,$$

where $\alpha-\beta=\gamma-\delta$, $\Re(\gamma-\alpha)>0$, $\Re(\frac{1}{2}-\alpha+\beta)>0$ and $|z|\neq 0$.

M. C. Gray (New York, N. Y.).

Meixner, J. Umformung gewisser Reihen, deren Glieder Produkte hypergeometrischer Funktionen sind. Deutsche Math. 6, 341–349 (1942). [MF 8609]

An elementary algebraic transformation of the functions occurring in the Pochhammer contour integral for the hypergeometric function provides a general means whereby a number of transformations between sums of products of hypergeometric series may be obtained. Typical of the results is the identity:

$$\sum_{n=0}^{\infty}\binom{k}{n}s^nF(-n,b;c;z)F(-n,\beta;\gamma;\zeta)$$
$$=(1+s)^k\sum_{n=0}^{\infty}\binom{k}{n}\left[\frac{z\zeta s}{(1+s)^2}\right]^n\frac{(b)_n(\beta)_n}{(c)_n(\gamma)_n}$$
$$\times F(n-k,b+n;c+n;sz/(1+s))$$
$$\times F(n-k,\beta+n;\gamma+n;s\zeta/(s+1)).$$

Several special cases and confluent forms are indicated. One application is given to orthogonal polynomials $P_n(x)$ belonging to generating functions of the type $f(t)\exp[xu(t)]$.

N. A. Hall (New Haven, Conn.).

Scheffé, Henry. Linear differential equations with two-term recurrence formulas. J. Math. Phys. Mass. Inst. Tech. 21, 240–249 (1942). [MF 8171]

On substituting a formal power series

$$w=\sum c_v(z-a)^v$$

in the differential equation

$$\sum_{j=0}^{n}p_j(z)w^{(j)},$$

$p_j(z)$ being analytic at a, a recurrence formula for the coefficients is obtained. This formula is called a two-term recurrence formula (TTRF) relative to the point a if it has only two terms

$$f(v)c_v+g(v)c_{v-h}=0,\quad h>0;\ v=h,\ h+1,\ \cdots.$$

The author gives the necessary and sufficient conditions on the coefficients of a differential equation in order that it have a TTRF. He then shows that such an equation can be transformed into a generalized hypergeometric equation. Finally the question whether a differential equation can have a TTRF relative to several distinct points is considered.

F. G. Dressel (Durham, N. C.).

Differential Equations

Piaggio, H. T. H. The operational calculus. Nature 152, 93 (1943). [MF 8853]

Zech, Th. Zum Abklingen nichtlinearer Schwingungen. Ing.-Arch. 13, 21–33 (1942). [MF 8596]

The author studies the monotone decreasing amplitude curves of the oscillatory solutions of the nonlinear differential equation (*) $y''+Ry'+a^2y+b^2y^3=0$, R, a^2 and b^2 real constants, by using energy considerations. Numerical examples and further generalizations of (*) are given.

A. E. Heins (Cambridge, Mass.).

Bulgakov, B. V. Maintained oscillations of automatically controlled systems. C. R. (Doklady) Acad. Sci. URSS (N.S.) 37, 250–253 (1942). [MF 8665]

The author considers an automatically controlled system described by the equations

$$\ddot\phi+M\dot\phi=-N\eta,\quad \dot\eta=F\left(\phi+\beta\dot\phi-\eta/a\right),$$

where F is the characteristic function of the servo-motor in the system and M, N, a and β are positive constants. The author then shows that such a system can execute maintained oscillations only if $M\beta<1$ and if $a>a_0$, where a_0 is the "threshold value" of a for maintained oscillations. An explicit formula is given for a. The author makes use of results obtained in an earlier paper. He has since published a translation of this paper [J. Franklin Inst. 235, 591–616 (1943); these Rev. 4, 245].

N. Levinson.

Friedrichs, K. O. and Stoker, J. J. Forced vibrations of systems with nonlinear restoring force. Quart. Appl. Math. 1, 97–115 (1943). [MF 8802]

This paper contains the substance of a lecture given by the authors at the Conference on Non-Linear Mechanics at Brown University in August, 1942. The authors discuss periodic solutions of

$$(*)\qquad \ddot x+f(x)=F\cos\omega t.$$

The method of approach is the point of interest in this paper, the results being largely well known. Unlike most treatments, here there is no vagueness or impression of arbitrary unjustifiable procedure to reach predetermined results. Moreover precision is combined with extreme simplicity so that the results obtained will be widely available. The authors indicate the effect of the nonlinearity of $f(x)$ in the relationship between the amplitude and frequency of the periodic solutions of (*). Several alternative procedures for computing this effect are given. Jump phenomena and subharmonic resonance are discussed as an indication of the considerable modification in physical performance in the non-linear case as compared with the familiar linear case. Stability and bifurcation are also considered.

N. Levinson (Cambridge, Mass.).

Müller, Max. Über die Existenz periodischer Lösungen bei gewissen Systemen gewöhnlicher Differentialgleichungen erster Ordnung. Math. Z. 48, 128–135 (1942). [MF 8565]

The author considers the system of differential equations

$$y_j'=f_j(x,y_1,y_2,\cdots,y_n),\qquad j=1,2,\cdots,n,$$

where the f_j have a common period in x. He develops a criterion which, if fulfilled, establishes the existence of at least one periodic solution for the system. This solution is obtained by a successive approximations procedure which is a generalization of a procedure of N. W. Adamoff for the case of a single first order differential equation.

N. Levinson (Cambridge, Mass.).

Niemytzki, V. Intégration qualitative approximative du système d'équations $dx/dt=Q(x, y)$, $dy/dt=P(x, y)$. C. R. (Doklady) Acad. Sci. URSS (N.S.) **38**, 62–65 (1943). [MF 8688]

This paper sketches a method for the discovery of periodic solutions of the system of differential equations given in its title. The case that such solutions do not exist can be recognized in a finite number of steps. If the process can be continued indefinitely, it shows the existence of periodic solutions. Use is made of a method of de la Vallée Poussin to find so-called "ϵ-solutions" by means of polygonal lines of Euler. For the way in which "ϵ-nets" are defined inside a family of arcs of integral curves, and an index of an ϵ-solution is defined, we must refer to the paper itself. One of the results establishes an upper boundary to the length of periodic solutions inside a given domain. *D. J. Struik.*

Haupt, Otto. Über Lösungen linearer Differentialgleichungen mit Asymptoten. Math. Z. **48**, 212–220 (1942). [MF 8548]

This paper is concerned with the solutions of a second order linear differential equation

$$\text{(L)} \qquad y''+g(x)y'+f(x)y=h(x),$$

where f, g, h are single-valued continuous functions on the interval $[a, +\infty)$. It is shown that a solution $y(x)$ of (L) possesses an asymptote (asymptotic line) if and only if y is expressible as

$$y(x)=x\left(c-\int_b^x\right)\phi(\tau)\tau^{-2}d\tau, \qquad b\geqq a>0,$$

where $\phi(x)$ is a function that is continuous on $[a, +\infty)$, is such that $\lim_{x\to+\infty}\phi(x)$ exists and is finite and which satisfies a certain integral equation of the second kind. The further results of this paper concern an equation (L) for which the functions f, g, h are continuous and $g(x)$, $xh(x)$ are absolutely Riemann integrable on $[a, +\infty)$. For an equation of this latter type the following results are established: (i) if $r(x)=xg(x)+x^2f(x)$ is absolutely integrable on $[a, +\infty)$, then every solution of (L) has an asymptote; (ii) if $r(x)$ is not integrable on $[a, +\infty)$, but is of constant sign for x sufficiently large, then whenever a solution of (L) has an asymptote this asymptote must be parallel to the x-axis; (iii) if $s(x)=g(x)+xf(x)$ is absolutely integrable on $[a, +\infty)$ and $r(x)\equiv xs(x)$ is not integrable on this interval, but $s(x)$ is of constant sign for x sufficiently large, then for b sufficiently large there exists a unique solution of (L) rendering a prescribed value to $y(b)-by'(b)$ and possessing an asymptote, which, in view of (ii), is parallel to the x-axis; (iv) if $s(x)$ is not integrable on $[a, +\infty)$, but is of constant sign for x sufficiently large, then whenever a solution of (L) has an asymptote this asymptote must be the x-axis. The results of this paper, when applied to the particular equation $y''+f(x)y=0$, contain as special cases the results of a recent paper by Bitterlich-Willmann [Monatsh. Math. Phys. **50**, 35–39 (1941)]. *W. T. Reid* (Chicago, Ill.).

Haupt, Otto. Über das asymptotische Verhalten der Lösungen gewisser linearer gewöhnlicher Differentialgleichungen. Math. Z. **48**, 289–292 (1942). [MF 8555]

The principal result of this paper is as follows: if $h(x)$, $g_\nu(x)$ ($\nu=0, 1, \cdots, n-1$) are continuous on $[a, +\infty)$, and $x^{n-1-\nu}g_\nu(x)$ ($\nu=0, 1, \cdots, n-1$) are absolutely integrable, while $h(x)$ is integrable (Riemann) on this interval, then for each solution $y(x)$ of the linear differential equation

$$L_n(y)\equiv y^{(n)}+g_{n-1}(x)y^{(n-1)}+\cdots+g_0(x)y=h(x)$$

the n functions

$$(n-1-\nu)!\,x^{\nu+1-n}y^{(\nu)}(x), \quad \nu=0, 1, \cdots, n-1,$$

converge to the same finite limit as $x\to+\infty$. The author first shows that, if $y^{(n-1)}(x)$ converges to a finite limit, then the remaining $n-1$ functions indicated above also converge to this limit as $x\to+\infty$. It is then shown that a solution y of $L_n(y)=h(x)$ is such that $\lim_{x\to+\infty}y^{(n-1)}(x)$ exists and is finite if and only if $y(x)$ is expressible in a prescribed integral fashion in terms of $L_{n-1}^*(y)=y^{(n-1)}(x)=\phi(x)$, and $\phi(x)$ in turn is a solution of a certain integral equation of the second kind such that $\lim_{x\to+\infty}\phi(x)$ exists and is finite. The paper concludes with some remarks on possible generalizations to the case of solutions of $L_n(y)=h(x)$ for which a given differential expression $L_{n-1}^*(y)$ of order $n-1$ converges to a limit as $x\to+\infty$. The consideration of solutions which possess asymptotic parabolas of degree $n-1$ involves the particular case

$$L_{n-1}^*(y)=x^nd^{n-1}[x^{-1}y(x)]/dx^{n-1},$$

which, for $n=2$, was considered in the paper reviewed above. The results of the present paper for the particular equation $y''+g_0(x)y=0$ include as a special case the results recently obtained by Caligo [Boll. Un. Mat. Ital (2) **3**, 286–295 (1941); these Rev. **3**, 119]. *W. T. Reid.*

Seifert, H. Zur asymptotischen Integration von Differentialgleichungen. Math. Z. **48**, 173–192 (1942). [MF 8567]

The paper is concerned with the differential equation

$$y''+p(x)y'-\{\rho^2q(x)+r(x)\}y=0,$$

in which ρ is an unbounded positive parameter, and primarily with the case in which the coefficients p, q, r are analytic functions which are real when x is real. If $q(x)$ has a simple zero, say at x_0, the solutions of the equation are familiarly of exponential and of oscillatory types on intervals of the axis of reals which lie on the one or the other side of x_0. The author singles out certain solutions which are real for real x, and derives the "connection formulas" which associate their asymptotic representations relative to a pair of fixed points $x_n<x_0$ and $x_P>x_0$. *R. E. Langer.*

Abdelhay, J. The existence of an oscillation theorem for a special differential equation of third order. Characteristic values. Anais Acad. Brasil. Ci. **14**, 385–409 (1942). (Portuguese) [MF 8440]

For the differential system (I): $[\theta(x)y']''=\lambda\phi(x)y$, $y(a)=y(b)=y'(b)=0$, where $\theta(x)$, $\phi(x)$ are real-valued functions of class $C^{(3)}$ on the finite interval $a\leqq x\leqq b$, the author proves the existence of a positive characteristic value λ such that the corresponding characteristic solution is positive on the open interval $a<x<b$. In particular, this characteristic value has the definitive property of being the least upper bound of the set of λ's for which the associated system (II): $[\theta(x)y']''=\lambda\phi(x)y$, $y(a)=y(b)=0$, $y'(b)=-1$, has a solution that is positive on the open interval $a<x<b$. Auxiliary to the proof of this result the author established various theorems on the behavior of solutions of systems (I) and (II) and the allied systems consisting of the boundary conditions of these systems and the nonhomogeneous differential equation $[\theta(x)u']''=\psi(x)$, where $\psi(x)\geqq0$ on the interval ab. *W. T. Reid* (Chicago, Ill.).

Kamke, E. Über die definiten selbstadjungierten Eigenwertaufgaben bei gewöhnlichen linearen Differentialgleichungen. IV. Math. Z. **48**, 67–100 (1942). [MF 8562]

This paper is concerned with the self-adjoint differential boundary problems

$$(1)\qquad F(y)=\lambda G(y),$$

$$(2)\qquad U_\mu(y)=0,\qquad \mu=1,\cdots,2m,$$

in which

$$F(y)=\sum_{\nu=0}^{m}(f_\nu y^{\nu})^{(\nu)},\quad G(y)=\sum_{\nu=0}^{n}(g_\nu y^{(\nu)})^{(\nu)}$$

with $f_\nu(x)$, $g_\nu(x)$ real and of class $C^{(\nu)}$; $f_m\neq 0$, $g_n\neq 0$, $0\leqq n<m$; and in which the forms

$$U_\mu(y)=\sum_{j=0}^{2m-1}\left[\alpha_{\mu j}y^{(j)}(a)+\beta_{\mu j}y^{(j)}(b)\right]$$

are real and linearly independent. In previous papers [Math. Z. **46**, 231–250, 251–286 (1940); these Rev. **2**, 52] the author showed that, if the relations

$$\int_a^b yF(y)dx\geqq 0,\quad \int_a^b yG(y)dx\geqq 0$$

maintain for every y which fulfills the conditions (2) and is of class $C^{(2m)}$, the problem

$$\left(\int_a^b yF(y)dx\right)\Big/\left(\int_a^b yG(y)dx\right)=\text{a minimum}$$

admits a solution if the domain of admissible functions is that comprised of functions of class $C^{(2m)}$ that fulfill the conditions (2) and $\int_a^b yG(y)dx>0$. The minimum in question is the smallest characteristic value, and the solving function the associated characteristic function.

The present paper is in the main devoted to showing that, under further hypotheses of "definiteness" on the boundary problem (more stringent than the conditions so designated in preceding papers), the domain of admissible functions in which the facts mentioned subsist may be enlarged, that is, the specifications may be relaxed.

R. E. Langer (Madison, Wis.).

Ritt, J. F. Bézout's theorem and algebraic differential equations. Trans. Amer. Math. Soc. **53**, 74–82 (1943). [MF 7825]

Let A and B be two algebraically irreducible differential forms in the unknowns y and z having respective orders m and n. Suppose that the general solutions of A and B have a non-vacuous intersection $\mathfrak{M}$ and let $\mathfrak{N}$ be an irreducible component of dimension zero of $\mathfrak{M}$. Then the author proves that if $m\leqq 1$, $n\leqq 1$ then the order of $\mathfrak{N}$ is $\leqq m+n$. An example is given to prove that the bound $m+n$ is not valid if one of the orders $n\geqq 4$.

N. Jacobson.

Maruhn, Karl. Zur eindeutigen Lösbarkeit der potentialtheoretischen Randwertaufgaben bei nichtbeschränkten Randwerten. Math. Z. **48**, 251–267 (1942). [MF 8552]

It is well known that the solutions of the boundary value problems for Laplace's equation are not unique when the boundary functions are allowed to assume infinite values. In the present paper the author considers what restrictions must be placed on the boundary functions and the solution functions to insure uniqueness, and proves the following theorem. Let $f(\sigma)$ be a function defined on the boundary S of a region T bounded by a smooth curve (or surface, in three dimensions), such that $\int_S|f|d\sigma<+\infty$, $f=\infty$ at a finite number of points P_i of S and f satisfies a Lipschitz condition in every closed set of S excluding the P_i. Then there exists at most one function u such that (1) u is harmonic in T and bounded in $T+S$, (2) $\partial u/\partial n$ tends uniformly to $f(\sigma)$ on every closed set of S excluding the P_i and

$$(3)\qquad \lim_{\delta\to 0}\int_{S_\delta}|\partial u/\partial n|d\sigma_\delta=\int_S|f(\sigma)|d\sigma,$$

S_δ being the parallel surface to S in T at distance δ along the normals. The method of proof consists in using the various limit assumptions in the hypotheses to show that, if u_1 and u_2 are two solutions and v their difference, then, for $\epsilon>0$ given, $\delta>0$ exists such that

$$\int_{S_\delta}|v\partial v/\partial n|d\sigma_\delta<\epsilon,$$

and hence, by an application of Green's formula, $v\equiv 0$. A similar theorem is proved for the boundary value problem in which $\partial u/\partial n+h(\sigma)u$ is assigned values on S, $h(\sigma)$ being a given nonpositive continuous function on S.

In the two dimensional case, the author is able, by making use of the conjugate potential function, to prove uniqueness under somewhat different assumptions. In this case, let T be a plane region bounded by a smooth curve S with arc length variable s, and let $f(s)$ be a function continuous and satisfying a Lipschitz condition except at points $s_1, s_2, \cdots, s_n$, where $f(s)$ satisfies inequalities of the form

$$|f(s)|<\text{const.}|s-s_\nu|^{-\lambda},\qquad 0<\lambda<1.$$

Then there exists at most one function u such that (1) u is harmonic in T and continuous in $T+S$, (2) $\partial u/\partial n\to f(s)$ uniformly except near the points s_ν, (3) on S, $u(s)$ satisfies a uniform Lipschitz condition and (4) if T is unbounded, $\partial u/\partial x$ and $\partial u/\partial y$ are $O(((x^2+y^2)^{1+\tau})^{-1})$ $\tau>0$. Attention is called to the fact that the solutions of these boundary value problems as potentials of single layer satisfy the various conditions of the above theorems, and consequently are the unique solutions mentioned.

J. W. Green.

Leonov, M. J. Properties of Green's spacial functions. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] (N.S.) **4**, 117–118 (1940). (Russian. English summary) [MF 7881]

Let g be a bounded domain of the x, y-plane, G its complementary domain. Let $D_g(D_G)$ be the domain of the three-dimensional x, y, z-space consisting of all points (x, y, z) for which $(x, y)\epsilon D_g((x, y)\epsilon D_G)$, $z=0$. Let $H_g^{(i)}(x, y, z, \xi, \eta, \zeta)$ $(H_G^{(i)})$ be the Green function of the ith kind of the domain $D_g(D_G)$, $i=1, 2$. The author shows that

$$H_g^{(1)}-(1/2r)+(1/2r_1)=\pm\{H_g^{(2)}-(1/2r)-(1/2r_1)\},$$

$$H_g^{(2)}-(1/2r)-(1/2r_1)=\pm\{H_g^{(1)}-(1/2r)+(1/2r_1)\},$$

where $r^2=(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2$, $r_1^2=(x-\xi)^2+(y-\eta)^2+(z+\zeta)^2$, and the upper (lower) sign holds for $z\zeta>0$ $(z\zeta<0)$.

L. Bers (Providence, R. I.).

Fedoroff, V. S. Sur les fonctions harmoniques. Rec. Math. [Mat. Sbornik] N.S. **12(54)**, 161–183 (1943). (Russian. French summary) [MF 8801]

Let B be a bounded domain in the three-dimensional Euclidean space, and let E be a perfect totally discontinuous set contained in B. Then (1) if each portion of E is of positive measure, there is a function $f(M)$ defined in B,

continuously differentiable there and harmonic in $B-E$; (2) the conclusion of (1) holds true for some sets E which are of measure 0; (3) for every function $f(M)$ satisfying the conclusion of (1) there is a sequence of functions $\varphi_n(P)$, each defined and bounded on E, and such that

$$f(Q)=\sum_{n=1}^{\infty}\int_E(\varphi_n(P)/r)d\tau+H(Q)$$

for $Q\epsilon B-E$; here $H(Q)$ is harmonic in B, and the series of integrals converges uniformly over any closed subset of $B-E$. Let $f(M)$ be a function continuous in B. The author introduces the operator

$$\Delta_\rho f(M)=15\rho^{-2}\left\{(4\pi\rho^2)^{-1}\int_S f(M')d\sigma-\tau^{-1}\int_K f(M')d\tau\right\},$$

where S and K denote the surface and the interior of the sphere of center M and radius ρ; $d\sigma$ and $d\tau$ denote the elements of area and of volume. It is shown that (4) $\Delta_\rho f(M)=\operatorname{div}\vec{U}_\rho(M)$, where

$$\vec{U}_\rho(M)=15(4\pi\rho^5)^{-1}\int_K f(M')\overrightarrow{MM'}d\tau,$$

and (5) a necessary and sufficient condition for a function $f(M)$ continuous in B to be harmonic there is that $\lim\Delta_\rho f(M)=0$ for every $M\epsilon B$ and $\rho\to 0$.

A. Zygmund (South Hadley, Mass.).

Wolf, František. On harmonic and analytic functions. Bull. Amer. Math. Soc. 49, 602–605 (1943). [MF 8861]

It is a well-known fact that, if the function $u(r,\theta)$ is harmonic in the circle $r<1$ and is there the Poisson integral of a summable function $f(\theta)$, then for any arc $\alpha<\theta<\beta$ of the circumference $r=1$ we have the decomposition $u=u_1+u_2$, where (1) u_1 and u_2 are both harmonic in $r<1$, (2) $u-u_1$ can be continued across the arc (α,β) and takes the value 0 on that arc, (3) u_2 may be continued across the complementary arc $\beta<\theta<2\pi+\alpha$ and vanishes along that arc. [It is enough to set $f=f_1+f_2$, where f_1 coincides with f along (α,β) and vanishes elsewhere, and to define u_i as the Poisson integral of f_i.] The author shows that this decomposition theorem is valid for the most general function $u(r,\theta)$ harmonic for $r<1$.

A. Zygmund.

Beckenbach, E. F. Conjugate harmonic functions. Duke Math. J. 10, 335–339 (1943). [MF 8472]

A set of continuous functions $x_j=x_j(u_1,\cdots,u_n)$, $j=1,\cdots,m$, is a conjugate harmonic set over a common region of definition if each x_j is harmonic and if for some function $\lambda(u_1,\cdots,u_n)$

$$\sum_{j=1}^{m}\frac{\partial x_j}{\partial u_k}\frac{\partial x_j}{\partial u_l}=[\lambda(u_1,\cdots,u_n)]\delta_{k,l},\quad k,l=1,\cdots,n,$$

where $\delta_{k,l}=1$ or 0 as $k=l$ or $k\neq l$. Let $r(u_1,\cdots,u_n)$ be the function $[\sum_{j=1}^m(x_j+a_j)^2]^{\frac12}$ with real parameters a_j, $j=1,\cdots,m$. Cioranescu proved [Bull. Sci. Math. (2) 56, 55–64 (1932)] for $m=n>2$ that the functions x_j form a conjugate harmonic set if and only if (a) each x_j is harmonic and (b) the function $-r^{2-n}$ is harmonic wherever $r>0$ for all real values of the parameters a_j. The author proves that for $m\geqq n>2$ the functions x_j form a conjugate harmonic set if and only if $-r^{2-n}$ is subharmonic for all real values of the parameters a_j. The proof shows that for $m=n$ the function $-r^{2-n}$ is harmonic where $r>0$. Cioranescu also discussed the case $m=n=2$, while the generalization $m\geqq n=2$ was treated by the author and Radó [Trans. Amer. Math. Soc. 35, 648–661 (1933)].

L. H. Loomis (Cambridge, Mass.).

Privaloff, I. Quelques remarques sur la théorie des fonctions subharmoniques. Rec. Math. [Mat. Sbornik] N.S. 12(54), 85–90 (1943). (Russian. French summary) [MF 8794]

A subharmonic function $u(x,y)$ can be considered to be a potential of a mass distribution. The mass in analytic domains is expressed by means of $u(x,y)$, whence the uniqueness of the mass distribution follows. It is also deduced that the set of points at which $u(x,y)=-\infty$ is (i) either non-dense or of second category and (ii) that it contains no linear continuum.

František Wolf.

Martin, Monroe H. The rectilinear motion of a gas. Amer. J. Math. 65, 391–407 (1943). [MF 8698]

Riemann [Mathematische Werke, Leipzig, 1892, pp. 156–178] reduced the problem of rectilinear motion of a gas to the partial differential equation

$$L(w)\equiv w_{rs}-aw_r+aw_s=0,$$

for which he gave a method of solution which is applicable to the general hyperbolic equation. The present paper gives another method of solution, apparently not applicable to the general case. The method parallels Riemann's with one essential change, namely that, instead of the operator adjoint to $L(w)$ used by Riemann, the author uses the "conjugate" operator:

$$M(v)\equiv v_{rs}+av_r-av_s,$$

which is so chosen that the expression

$$(v_r-v_s)L(w)+(w_r-w_s)M(v)$$

is the divergence of the vector $[-w_sv_s, w_rv_r]$. The problem is reduced to the solution of $M(v)=0$ with certain special boundary conditions. As an application the adiabatic case is studied in detail and one example with particular initial conditions is given to illustrate the meaning of the results.

W. Kaplan (Ann Arbor, Mich.).

Garnea, E. G. On a new application of Jacobi polynomials in connection with the mean value theorem. Bull. Amer. Math. Soc. 49, 541–548 (1943). [MF 8854]

Let $U(y_1,y_2,\cdots,y_r)$ be a polyharmonic function of order $2m$, that is, $\Delta^{(2m)}U=0$, where $\Delta^{(2m)}$ is the Laplacian operator applied successively $2m$ times. Assume that (*) $\int_{S_r}Ud\tau=0$, integration extended over the volume of a hypersphere of radius R in the r-dimensional space. An incomplete statement of Gioranescu [Mathematica, Cluj 9, 184–193 (1935)] is as follows: under (*) there exists at least one point P_0 inside the hypersphere of radius x_MR such that $U(P_0)=0$, where x_M is the largest of the solutions x_i of the system of equations

$$(**)\qquad \sum_{i=1}^{m}k_ix_i^{2q}=r/(r+2q),$$

with the unknowns k_i, x_i $(0\leqq q\leqq 2m-1)$, provided all k_i are positive and the x_i are real and lie inside $(-1,1)$ (similarly in case the order of U is odd). In the present paper the statement of Cioranescu is completed by showing that the system (**) always has a solution $\{k_i,x_i\}$ satisfying the above requirements. The proof follows Shohat's method based upon very simple properties of orthogonal polynomials, more precisely: we identify (**) with fundamental relations in the theory of Gaussian quadrature

applied to Jacobi polynomials. The particular case $r=1$ yields the precise form of the mean-value theorem as given by Montel, Tchakaloff, Biernacki. *J. Shohat.*

John, Fritz. Linear partial differential equations with analytic coefficients. Proc. Nat. Acad. Sci. U. S. A. **29**, 98–104 (1943). [MF 8209]

This paper mentions some significant results in the Cauchy problem for linear nonhomogeneous mth order partial differential equations in n independent variables. The coefficients and data surface are assumed analytic. No proofs are given but some integral identities used in the writer's demonstrations are stated. Generality is attained by the natural expedient of expressing the theorems in terms of properties of the characteristics so that the usual division into types is implicitly covered by noting the corresponding special nature of the characteristics. The interest is in uniqueness theorems, functional character of the solution and characterization of manifolds (using Hadamard's terminology) for which the problem is not correctly set. The results seem quite plausible. Thus a domain of uniqueness for a solution determined by an admissible manifold is obtained by analytic deformation, with fixed boundaries, of the data surface under the requirement that characteristic normals are avoided. If some two dimensional plane through the normal to the data surface at a point contains no characteristic direction, the Cauchy problem is not soluble locally, in general. Another result extends the classical Picard-Bernstein theorem on the analyticity of solutions of elliptic equations with analytic coefficients.

D. G. Bourgin (Urbana, Ill.).

Azevedo do Amaral, Ignacio M. On the integration of ordinary and partial linear differential equations and on the solution of integral equations of the first kind in the cases either of functions of one or of several variables. Anais Acad. Brasil. Ci. **14**, 293–303 (1942). (Portuguese) [MF 8439]

In the present paper the author redevelops the results he obtained previously [same Anais **13**, 305–317 (1941); **14**, 83–86 (1942); these Rev. **3**, 243; **4**, 99] on the integration of ordinary and partial differential equations and integral equations of the first kind of Fredholm and Volterra types. This paper is characterized by the formality that pervaded the author's previous papers on this topic; in particular, it is not clear as to what classes of equations are amenable to his very special methods. *W. T. Reid* (Chicago, Ill.).

Theoretical Statistics

Kaplansky, Irving. A characterization of the normal distribution. Ann. Math. Statistics **14**, 197–198 (1943). [MF 8781]

R. A. Fisher gave a geometric derivation of the joint distribution of mean and variance in samples from a normal population [Metron **5**, 90–104 (1925)]. On examining the argument, however, we find that an (apparently) more general result is actually established: if $f(x_1)\cdots f(x_n)$ is a function $g(m, s)$ of the sample mean m and standard deviation s, then the probability density of m and s in samples of n from the population $f(x)$ is $g(m, s)s^{n-2}$. This condition on $f(x)$ is of course satisfied if $f(x)$ is normal; in this note we shall conversely show that for $n\geqq 3$ it characterizes the normal distribution. *From the introduction.*

Shannon, S. Comparative aspects of the point binomial polygon and its associated normal curve of error. Record. Amer. Inst. Actuar. **31**, 208–226 (1942). [MF 8319]

This is a discussion of S. Shannon's paper printed in Record. Amer. Inst. Actuar. **27**, 372–397 (1938). The main contribution [pp. 208–222] is by C. A. Spoerl, who discusses the approximation of the normal curve to the binomial distribution and various problems of numerical computations connected with it. *W. Feller* (Providence, R. I.).

Samuelson, Paul A. Fitting general Gram-Charlier series. Ann. Math. Statistics **14**, 179–187 (1943). [MF 8778]

If $f(x)$ is a "parent" frequency function with moments and derivatives of all orders and high order contact at the extremities of the distribution, and if $F(x)$ is an arbitrary frequency curve the expansion

$$F(x)=a_0f(x)+a_1f'(x)+a_2f''(x)+\cdots \tag{1}$$

may be called a generalized Gram-Charlier expansion. Equating moments and integrating by parts in (1) we obtain the formal relations

$$L_n(F)=\sum_{j=0}^{n} L_{n-j}(f)(-1)^j a_j, \tag{2}$$

where $n!L_n(F)$ denotes the nth moment of F. Equations (2) are readily solved in sequence for the a's. The author compares this method with Carver's method of expanding both F and f in terms of the normal frequency function, and with Charlier's method of orthogonal polynomials. He discusses the moment generating function, treats expansion (1) when derivatives are replaced by differences, and comments on convergence. *W. E. Milne* (Corvallis, Ore.).

Nair, K. R. and Shrivastava, M. P. On a simple method of curve fitting. Sankhyā **6**, 121–132 (1942). [MF 8634]

A function $y=f(x;\alpha_1, \cdots, \alpha_k)$ is to be fitted to the empirical data $\{(x_1, y_1), \cdots, (x_n, y_n)\}$. The n residuals

$$v_i=y_i-f(x_i;\alpha_1, \cdots, \alpha_k), \qquad i=1, 2, \cdots, n,$$

are formed. The authors use the method of group averages to estimate the unknown parameters. The residuals are arranged into k groups with p_j residuals in the jth group. The k parameters are estimated by solving the system of linear equations $\sum_{i=1}^{p_j} v_i=0$, $j=1, 2, \cdots, k$. The sum $p_1+p_2+\cdots+p_k\leqq n$, so that some of the residuals may be omitted. Assuming that the residuals are normally distributed about zero with the same variance, the maximum likelihood estimate reduces to the method of least squares. The authors assume f to be a polynomial and aim to find "best values" for the numbers $p_1, p_2, \cdots, p_k$. "Best values" are defined as the values of $p_1, p_2, \cdots, p_k$ which will maximize the efficiencies of the estimates of the parameters in comparison to the estimates obtained by the method of least squares. It is impossible to maximize simultaneously the efficiency of the estimates of all the parameters. Therefore the authors determine the numbers $p_1, p_2, \cdots, p_k$ so as to maximize the efficiency of the coefficient of the highest degree. By this method straight lines and parabolas are fitted to the observations and the efficiencies of the resulting estimates are discussed. *E. Lukacs* (Jacksonville, Ill.).

Gebelein, Hans. Bemerkung über ein von W. Höffding vorgeschlagenes, massstabsinvariantes Korrelationsmass. Z. Angew. Math. Mech. **22**, 171–173 (1942). [MF 8904]

Gumbel, E. J. On serial numbers. Ann. Math. Statistics **14**, 163–178 (1943). [MF 8777]

The author begins by observing that, if n observations of a continuous random variable be arranged in order of magnitude, for the purpose of comparison of the sample with the theoretical ogive for the universe, the mth one x_m may equally logically be assigned the rank, or cumulative frequency, m or $m-1$. The same difficulty exists in the use of the equiprobability test or in the comparison of return periods. From the distribution function of the mth value x_m he obtains the equation satisfied by the most probable value $\tilde{x}_m$ of x_m, which of course depends upon the distribution law for the universe. This equation is used in two ways. First, if m be known, an adjusted frequency is assigned to $\tilde{x}_m$, which is $m-\frac{1}{2}$ plus a correction, which is now made unique. Second, if the adjusted frequency is supposed known, then the most probable value of the mth observation is given which gives estimates of the grades (percentiles) from the serial numbers. Both of these applications are discussed and illustrated. For large samples the standard errors of these estimated grades are given. An interesting application is the determination of the "most precise grade." The method outlined for the comparison of a sample with a theoretical ogive is numerically illustrated. *C. C. Craig.*

Schneider, Otto. On a parameter used to characterize bivariate statistical distributions. An. Soc. Ci. Argentina **133**, 397–401 (1942). (Spanish) [MF 8428]

Letting θ be the angle of the major axis of the ellipse defined by the exponent of a normal bivariate distribution, the asymptotic formula for the variance of $\tan 2\theta$ is evaluated by the δ-method. *W. G. Madow.*

Hoel, Paul G. On indices of dispersion. Ann. Math. Statistics **14**, 155–162 (1943). [MF 8776]

For a large expected number m of successes it is known that the binomial index of dispersion is distributed as χ^2 with $N-1$ degrees of freedom, N being the sample size. The present paper is devoted to an investigation of the adequacy of the χ^2-distribution for small samples both for the binomial index and for the Poisson index. The first four sampling moments of these indices are found correct to terms in N^{-3} and these moments are compared with the corresponding moments of the χ^2-distribution. For $m\geqq 5$ it appears that the χ^2-approximation is very satisfactory in both cases but for smaller values of m for the binomial index it is less so. As the author remarks the numerical coefficients neglected in getting the above sample moments to terms in N^{-3} are large and the reviewer's experience in similar calculations is that for small samples the results may indeed be "of questionable accuracy." *C. C. Craig.*

Cochran, W. G. The χ^2 correction for continuity. Iowa State Coll. J. Sci. **16**, 421–436 (1942). [MF 8022]

The following practical rule is given for obtaining a correction of χ^2 for continuity: calculate χ^2 by the usual formula, then find the next possible value of χ^2 lower than the one to be tested, and use the tabular probability for a value of χ^2 midway between the two. The correction for continuity indicated by Yates follows from the preceding rule if one applies it to the particular cases discussed by Yates; in other cases, however, the general rule may lead to different corrections. The use of the correction is illustrated by means of numerical examples. The last section of the paper contains a discussion of the effect on the χ^2-distribution of one class with a small expectation. One of the interesting results is that, in the case of only one small-expectation class, the conventional lower bound 5 for that expectation is much too restrictive. For example, for 6 or more degrees of freedom the tabulated χ^2-distribution gives a good approximation at the 1 per cent level if that expectation is as small as 2. *Z. W. Birnbaum.*

Craig, Allen T. A note on the best linear estimate. Ann. Math. Statistics **14**, 88–90 (1943). [MF 8252]

The purpose of the note is to show by an example that, y and y' being two unbiased estimates of the same parameter θ, both belonging to the same class, of which y is the "best" estimate, having the least standard error, it may happen that

$$P\{|y-\theta|\leqq d\}\leqq P\{|y'-\theta|\leqq d\}$$

for all $d>0$. Let $x_1, x_2, \cdots, x_n$ be a sample of n independently observed values of a random variable X, ordered so that $x_i\leqq x_{i+1}$, $i=1, 2, \cdots, n-1$. The class C of estimates considered is that of linear functions $\sum c_i x_i$ with arbitrary constant coefficients c_i. If X is uniformly distributed within an interval from zero to $2\theta>0$, where θ is the parameter to be estimated, then it is found that the best unbiased estimate of class C is $y=(n+1)x_n/2n$. This estimate y is compared with another $w=(x_1+x_2)/2$ which, according to a result attributed to Pittman, is expected to have the property that, whatever $d>0$ and whatever any other estimate T of θ based on the same sample, including $T=y$,

$$P\{|T-\theta|<d\}\leqq P\{|w-\theta|<d\}.$$

There seems to be somewhere some misunderstanding concerning this result. In fact, using the actual distributions of y and w given by the author it is easy to check that, contrary to the assertion,

$$P\{|w-\theta|<d\}<P\{|y-\theta|<d\}$$

for all values of $d>0$, except those within an interval $0<d<a<\theta/n$. *J. Neyman* (Berkeley, Calif.).

Paulson, Edward. A note on the estimation of some mean values for a bivariate distribution. Ann. Math. Statistics **13**, 440–445 (1942). [MF 7903]

Let x and y be correlated variates of which it may be assumed that (a) σ_x^2, σ_y^2, ρ_{xy} are known a priori or that (b) they are not known. The following two problems are discussed under either of the assumptions (a) and (b). Problem 1: to set up confidence limits for the ratio m_x/m_y of the expectations of x and y, a random sample (x_i, y_i), $i=1, 2, \cdots, n$, being available. Problem 2: to estimate m_x if such a random sample is available and m_y is known a priori. The discussion is based on the assumption that the joint probability density $f(x, y)$ is normal but it is pointed out that, for sufficiently large samples, the statements derived in the paper also hold approximately in the non-normal case. *Z. W. Birnbaum* (Seattle, Wash.).

Paulson, Edward. A note on tolerance limits. Ann. Math. Statistics **14**, 90–93 (1943). [MF 8253]

The relationship between tolerance limits and confidence limits is pointed out. It is shown that tolerance limits may be derived by determining confidence limits, not for a parameter of the distribution, but for a future random observation (or for some function of the observations in a future sample). This method is used to establish tolerance limits for a pair of jointly normally distributed variates x

and y on the basis of a sample of n observations. The tolerance region obtained is the interior of an ellipse in the (x, y) plane. The shape and location of the ellipse are, of course, functions of the n observations on x and y.

A. Wald (New York, N. Y.).

Moore, Geoffrey H. and Wallis, W. Allen. Time series significance tests based on signs of differences. J. Amer. Statist. Assoc. 38, 153–164 (1943). [MF 8435]

This paper is a continuation of a line of attack begun by the authors in an earlier paper [National Bureau of Economic Research, Technical Paper no. 1, New York, 1941; see these Rev. **3**, 176] which discusses three additional tests based on the signs of first differences. The first is based on the number of signs of plus and minus differences. For up to 12 observations the exact sampling distributions are tabulated; for more than 12, from the behavior of the third and fourth moments it is judged that the normal curve of error will be satisfactory, using the mean and variance given by the authors. The second test is based on the tendency for like signs to cluster together within a period. The required probabilities can be found from a recursion formula given in the earlier paper; here results are tabulated in one special case. The third of these tests applies to the comparison of two series by means of a 2×2 table of signs of first differences. Here the sampling distribution is known only on the hypothesis of randomly arranged first differences.

C. C. Craig (Ann Arbor, Mich.).

Kishen, K. On expressing any single degree of freedom for treatments in an s^m factorial arrangement in terms of its sets for main effects and interactions. Sankhyā 6, 133–140 (1942). [MF 8635]

Ramamurti, B. and Sitaraman, B. On maximal sets of confounded interactions in a $(2^n, 2^k)$ confounded design. Sankhyā 6, 183–188 (1942). [MF 8638]

Bhattacharyya, A. A note on Ramamurti's problem of maximal sets. Sankhyā 6, 189–192 (1942). [MF 8639]

Nair, K. R. Efficiency of the adjustment for concomitant characters in biological experiments. Sankhyā 6, 167–174 (1942). [MF 8636]

Jeming, Joseph. Estimates of average service life and life expectancies and the standard deviation of such estimates. Econometrica 11, 141–150 (1943). [MF 8196]

Let y_x be the value of a plant and Δy_x the retirement from this plant during one year. It is assumed that values of y_x and Δy_x are observed at N different time points. The retirement ratio is defined as $\Delta y_x/y_x$; let $\zeta_x=E(\Delta y_x/y_x)$ be the population regression of the retirement ratio and z_x the least square estimate of ζ_x. The theoretical life table is then defined by the relations $\eta_0=1$ and $(\eta_x-\eta_{x+1})/\eta_x=\zeta_x$. Similarly the smoothed life table is defined by the relations $Y_0=1$, $(Y_x-Y_{x+1})/Y_x=z_x$. The expected value of the length of life $\lambda=\int_0^\infty\eta_x dx$ is estimated by $L=\int_0^\infty Y_x dx$. Assuming that the regression of the retirement ratio is a linear function of time, the author expands L in a Taylor series. Neglecting the terms of second and higher order, he finds an approximation for the standard deviation of L valid for large N. Furthermore, he discusses the standard deviation of the life expectancy at age t as well as some special cases.

E. Lukacs (Jacksonville, Ill.).

Lang, Kermit. Analysis of net premium formulas for the income endowment policy. Record. Amer. Inst. Actuar. 31, 398–405 (1942). [MF 8323]

Greville, T. N. E. "Census" methods of constructing mortality tables and their relation to "insurance" methods. Record. Amer. Inst. Actuar. 31, 367–373 (1942). [MF 8322]

Leser, C. E. V. The consumer's demand for money. Econometrica 11, 123–140 (1943). [MF 8195]

In the theory of consumer's demand it is usually assumed that the consumer's total expenditure is equal to his income. In this paper no such assumption is made, and money is introduced into the utility function of the consumer in the following manner. Let there be n commodities and denote by $p_1, \cdots, p_n$ their prices. Suppose that the consumer buys the quantities $x_1, \cdots, x_n$ during his budget period and that at the end of the period he is still in possession of the amount H in cash. Then $x_i'=H/p_i$ is the quantity of the ith commodity which he will be able to buy at any time in the future. Thus the total utility of consumption and holdings of money is determined by the quantities $x_1, \cdots, x_n$, $x_1', \cdots, x_n'$, as distinct from the classical theory where the utility is a function of $x_1, \cdots, x_n$ only. The utility function u, as in the classical theory, is introduced merely for the sake of convenience, since the ultimate results depend only on the indifference map implied by the utility function.

In the first part of the paper the author derives the optimum conditions corresponding to the utility function

$$u(x_1, \cdots, x_n, x_1', \cdots, x_n')$$

and discusses the effects on demand of variations in income and prices. In the second part the author deals with the effects of savings on the consumer's behavior at a future date.

A. Wald (New York, N. Y.).

NUMERICAL AND GRAPHICAL METHODS

★Jahnke, Eugene and Emde, Fritz. Tables of Functions with Formulae and Curves. Dover Publications, New York, 1943. 303 pp.+addenda 76 pp. $3.50.

The present edition contains all material of the 1938 edition [Teubner, Leipzig] and, in addition, a 76 page section of "Tables of Elementary Functions" from the 1933 edition. There exists a later German edition [cf. these Rev. **3**, 152] of the book, but copies of it do not seem to have reached this country. The text is in English and German.

W. Feller (Providence, R. I.).

★Stratton, J. A., Morse, P. M., Chu, L. J. and Hutner, R. A. Elliptic Cylinder and Spheroidal Wave Functions, Including Tables of Separation Constants and Coefficients. John Wiley and Sons, Inc., New York, 1941. xii+127 pp. $1.00.

The first 50 pages form an article by Chu and Stratton [J. Math. Phys. Mass. Inst. Tech. **20**, 259–309 (1941); these Rev. **3**, 116]. The next 20 pages of diagrams and formulae followed by two pages of notes were written by Hutner to form an introduction to the tables occupying the last 50

* Contrary to the statement in the review there does not seem to exist a later German edition of the book. The book referred to is by Emde and contains only tables of elementary functions.

odd pages. For the elliptic cylinder functions the angular and radial functions are classified into even and odd types and the appropriate formulae are given, namely, the orthogonal relations, expansions in Bessel functions, special values of the radial functions and the value of the Wronskian. A similar plan is used for the spheroidal functions of the prolate and oblate types. Series of Legendre functions are used for the angular solutions and series of Bessel functions of half a negative integer for the radial solutions. Five significant figures are given in the tables of coefficients but the last is considered doubtful. The list of errata given on the included slip apply only to the two first parts. The tables must, of course, be used in conjunction with others listed in the notes. It is to be hoped that those which are not yet completed will be soon available so that full use can be made of the wealth of mathematical results which will follow from a complete numerical grasp of the properties of the functions under consideration. *H. Bateman.*

★Miscellaneous Physical Tables. Planck's Radiation Functions and Electronic Functions. Prepared by the Federal Works Agency, Work Projects Administration for the City of New York, as a Report of Official Project No. 65-2-97-33; conducted under the sponsorship of the National Bureau of Standards. Technical Director: Arnold N. Lowan. New York, 1941. vi+58 pp. $1.50.

These tables are in two parts (Tables I–IV and Table V). In the first part the functions which are tabulated are the Planck radiation functions

$$\mathcal{R}_\lambda = c_1\lambda^{-5}(e^{c_2/(\lambda T)}-1)^{-1}, \qquad \mathcal{R}_{0-\lambda} = \int_0^\lambda \mathcal{R}_\lambda d\lambda,$$

$$N_\lambda = 2\pi c\lambda^{-4}(e^{c_2/(\lambda T)}-1)^{-1}, \quad N_{0-\lambda} = \int_0^\lambda N_\lambda d\lambda,$$

with $c=2.99776\times10^{10}$, $c_1=3.732\times10^{-5}$ and $c_2=1.436$. These functions are tabulated to five significant figures with arguments λT (Table I) and λ for $T=1000°\mathrm{K}$ (Table II) together with their first and second central differences. An additional table (III) provides the function N_λ for various values of T in the range 1000°K–6000°K with argument λ. Finally there is table IV which would enable corrections to be incorporated in the principal tables if small changes in the values of the constants c, c_1 and c_2 should eventually appear necessary. The tables are conveniently arranged and as the Planck functions are of frequent occurrence in a variety of physical and astronomical calculations both astronomers and physicists will gladly welcome them.

In the second part (Table V) the functions

$$G=(1-\beta^2)^{-\frac{1}{2}},\ \beta G,\ V=10^{-8}(m_0/e)c^2(G-1),\quad H\rho=(m_0/e)c\beta G$$

(where e/m_0 is the specific charge of the electron and c the velocity of light) are tabulated with suitably "graded" intervals in the argument β. These functions are of common occurrence in physical calculations in which the relativistic mass variation with velocity is an important factor. Consequently, these tables will be of particular use to the nuclear and the cosmic-ray physicists.

S. Chandrasekhar (Williams Bay, Wis.).

Lowan, Arnold N. and Abramowitz, Milton. Table of the integrals $\int_0^x J_0(t)dt$ and $\int_0^x Y_0(t)dt$. J. Math. Phys. Mass. Inst. Tech. 22, 2–12 (1943). [MF 8766]

This table gives the values of the above integrals to 10 places of decimals over the range $x=0$ to $x=10$ at intervals of 0.01. *W. E. Milne* (Corvallis, Ore.).

The paper was quoted from a reprint. The correct pagination is 219–229.

Hall, Newman A. The solution of the trinomial equation in infinite series by the method of iteration. Nat. Math. Mag. 15, 1–11 (1941). [MF 8180]

It is shown that the equation

$$z^{m+n}-z^m+a=0$$

can be completely solved by iterating the relation

$$z_{k+1}=\lambda(1-\mu z_k{}^r)^{1/s},$$

where λ is an appropriate root of unity, μ depends on a, and r and s depend on m and n in a simple manner. These iterates and their powers are, in fact, the partial sums of certain powers series in $\mu\lambda^r$, whose coefficients can be computed explicitly and which have been previously obtained by other methods. The regions of validity of the iteration process are established by determining the regions of convergence of these series. A bibliography of 26 papers is included. *P. W. Ketchum* (Urbana, Ill.).

Spoerl, Charles A. Solving equations in the machine age. Record. Amer. Inst. Actuar. 31, 129–149, 490–506 (1942). [MF 8318, 8324]

To solve $x=\phi(x)$ the author uses the iterative scheme $x_{i+1}=(1-a)x_i+a\phi(x_i)$, where a is a parameter to be chosen so that $1-a+a\phi'(x)$ becomes small when x is near to the true root [cf. von Mises and Geiringer, Z. Angew. Math. Mech. 9, 58–77 (1929)]. Examples and detailed instructions for the computor are given. The author suggests that the method be used for inverse interpolation. The second part [pp. 490–506] contains a discussion contributed by various authors. *W. Feller* (Providence, R. I.).

Cornock, A. F. and Hughes, Joan M. The evaluation of the complex roots of algebraic equations. Philos. Mag. (7) 34, 314–320 (1943). [MF 8631]

To find a (single) root of an algebraic equation $f(z)=0$ the authors propose to transfer the origin to a suitable point x_0 on the real axis and then, using Horner's method, to evaluate $f(z)$ for a series of values x_0+iy in order to find a point x_0+iy_0 for which $|f^2|$ is a minimum. Then, the origin is transfered to x_0+iy_0 and the process repeated. The initial value x_0 is selected near a minimum of $\Re f(z)$. *W. Feller.*

Levi, B. On the approximate solution of transcendental equations represented by Taylor series. Math. Notae 3, 1–40 (1943). (Spanish) [MF 8529]

This paper is devoted to a study of methods of solving transcendental equations expressed in the form

$$f(x)=1+a_1x+a_2x^2+a_3x^3+\cdots=0.$$

The first method considered is the one usually known as Newton's method. The second is Hadamard's method in which one determines the radius of convergence of the power series for the reciprocal of $f(x)$, and thus finds the modulus of the root (or roots) nearest the origin. The third is an extension to the transcendental case of the procedure commonly known as Graeffe's root-squaring method. Since in calculations with power series one actually operates on a partial sum (that is, a polynomial), certain pitfalls may arise in applying purely algebraic methods to the transcendental case. (For instance $e^z=0$ has no finite root while the equation using the nth degree partial sum of the series has n finite roots.) Criteria are given in the paper to assist in diagnosing such cases.

For sake of illustration and comparison each method is used to find the first root of $J_0(x)=0$. (The third method

is applied to find the first three roots.) The third method is also applied to the equations $e^z=0$ and $z^{-1}\log(1+z)=0$, and brings out the fact that they have no roots in the circle of convergence. *W. E. Milne* (Corvallis, Ore.).

Boyer, John. Osculatory interpolation in practice. Record Amer. Inst. Actuar. **31**, 337–350 (1942). [MF 8321]

The author expresses certain osculatory interpolation formulas (the King-Karup formula and the Jenkins formulas) in terms of ordinates rather than differences and enumerates some practical advantages of so doing. A special discussion is given for end points. Illustrative examples are worked. *W. E. Milne* (Corvallis, Ore.).

Feller, Willy. On A. C. Aitken's method of interpolation. Quart. Appl. Math. **1**, 86–87 (1943). [MF 8191]

A simplification of Aitken's [Proc. Edinburgh Math. Soc. (2) **3**, 56–76 (1932); Proc. Roy. Soc. Edinburgh. Sect. A. **58**, 161–175 (1938)] method of interpolation is presented which applies to an odd as well as to an even number of data. The method is one of iteration and does not use either differences or tables of interpolation coefficients. It is well adapted to machine computation. From $n+1$ given data one obtains n values such that a polynomial of degree $n-1$ through these values will be equal at the required point to an nth degree polynomial through the original data. Regarding these n values as new data, the process may be repeated; after n repetitions the single number left is the desired interpolated value. The given data need not be equally or even symmetrically spaced, but, if such is the case, suitable modifications of the method enable one to reduce the number of values more rapidly than one at a time.

P. W. Ketchum (Urbana, Ill.).

Berjman, Elena. A solution of the problem of least square adjustment by Gauss polynomials, using the computation and tabulation of the parametric coefficients of parabolic functions from the first to the fifth order for series having up to one hundred elements. An. Soc. Ci. Argentina **132**, 34–48, 104–117, 212–217 (1941); **133**, 208–215, 442–445 (1942). (Spanish) [MF 6085]

If a polynomial $y=F_n(x)$ of degree n is fitted by the method of least squares to a set of values (x_i, y_i), $i=1, 2, \cdots, N$, where the x_i are equally spaced, it is known that the coefficients c_m of x^m, $m=0, 1, \cdots, n$, in $F_n(x)$ are linear combinations of the moments

$$\lambda_k=\sum x_i^k y_i, \qquad k=0, 1, \cdots, n,$$

with coefficients $A_{m,k}$ which are functions of n and N alone. The author derives the functions F_1, F_2, F_3, F_4, F_5 in careful detail, and gives tables of all the coefficients $A_{m,k}$ for $n=1$ to 5 and for $N=n+1$ to 100. These tables are printed to 8 significant figures. For convenience in calculating the λ_k a table of 2nd, 3rd, 4th and 5th powers of integers to 50, and half odd integers to 49.5 is appended. Methods of using the tables are carefully illustrated with numerical examples.

W. E. Milne (Corvallis, Ore.).

Walther, A. und Brinkmann, K. Zum Sprungstellen-Verfahren, insbesondere für die Entwicklung nach Kugelfunktionen. Ing.-Arch. **13**, 1–8 (1942). [MF 8594]

The authors devise a notation and several formulas to aid in the expansion in series of Legendre polynomials of functions made up of segments of polynomials of low degree, particularly "block" functions, and broken line functions. Illustrative numerical examples are worked in detail.

W. E. Milne (Corvallis, Ore.).

Bödewadt, U. T. Ein vereinfachtes Interpolationsverfahren. Z. Angew. Math. Mech. **20**, 361–363 (1940). [MF 8674]

The author describes a simple method for subtabulation by twentieths, using first and second differences, valid if the third difference does not exceed 64 in units of the last decimal retained. The procedure is as follows. Put the initial value y_0 in the calculating machine. Put $D_1=\Delta y_0/20-(\Delta^2 y_0+\Delta^2 y_{-1})/100$ in the keyboard and add four times. Then add $(\Delta^2 y_0+\Delta^2 y_{-1})/200$ to D_1 in the keyboard and add four more times. Add the same correction again in the keyboard and add four times. Proceed by fours in this manner until y_1 is reached. *W. E. Milne.*

Massera, José L. Formulae for finite differences with applications to the approximate integration of differential equations of first order. Publ. Inst. Mat. Univ. Nac. Litoral **4**, 99–166 (1943). (Spanish) [MF 8528]

This paper undertakes a study and comparison of various methods for the numerical solution of differential equations of the first order. As a basis for the study, the author presents a systematic collection of formulas, including the Newton-Cotes open and closed quadrature formulas and various other analagous formulas useful for numerical integration. A detailed study of the error terms for these formulas is given, with particular application to the solution of differential equations. His conclusion is that the best methods for the numerical solution of differential equations are those employing symmetric quadrature formulas such as Simpson's rule and similar formulas of higher degree. Numerical examples are supplied for illustration.

W. E. Milne (Corvallis, Ore.).

Collatz, L. Berichtigung zu der Arbeit: "Vergleich der Integralgleichungsmethode von Bucerius mit dem Ritzschen Verfahren" in Astron. Nachr. 271.116. Astr. Nachr. **272**, 77 (1941). [MF 8576]

The paper appeared in Astr. Nachr. **271**, 116–120 (1941); these Rev. **3**, 154.

Kappus, R. Graphische Lösung der Differentialgleichung $Y''+\varphi(x)\cdot Y=f(x)$ bei beliebigen Anfangs-und Randbedingungen. Z. Verein. Deutsch. Ingenieure **87**, 26–28 (1943). [MF 8333]

Stange, K. Zur Berechnung einer Flugbahnschar nach dem Athenschen Verfahren. Z. Angew. Math. Mech. **20**, 350–357 (1940). [MF 8670]

Athen [Z. Angew. Math. Mech. **19**, 361–371 (1939)] presented an interpolation procedure for constructing a family of ballistic trajectories for varying angle of departure φ, muzzle velocity and ballistic coefficient being constant. He referred the trajectory to an oblique y, s coordinate system with s axis along the line of departure and y axis vertically downward. Then y and s and their first time derivatives are represented by convergent power series in $\lambda=(1+\sin\varphi)/2$, with coefficients which are independent of φ. Having obtained two or three trajectories by numerical calculation he used the leading terms of the power series as interpolating formulae. In the paper under review Stange modifies this procedure so as to make it apply directly to the customary rectangular coordinates (X, Y) of a trajectory. He shows that if λ^2 and higher powers are neglected the coordinates at the same instant on three trajectories (X_1, Y_1), (X_2, Y_2), (X_3, Y_3) corresponding to three different elevations φ_1, φ_2, φ_3 are connected by the

equations

$$\frac{\sin\varphi_2-\sin\varphi_3}{\cos\varphi_1}X_1+\frac{\sin\varphi_3-\sin\varphi_1}{\cos\varphi_2}X_2+\frac{\sin\varphi_1-\sin\varphi_2}{\cos\varphi_3}X_3=0$$

and

$$\begin{aligned}(\sin\varphi_2-\sin\varphi_3)Y_1+(\sin\varphi_3-\sin\varphi_1)Y_2+(\sin\varphi_1-\sin\varphi_2)Y_3\\+(\sin\varphi_3-\sin\varphi_1)(\sin\varphi_2-\sin\varphi_3)\\\times\{X_1/\cos\varphi_1-X_2/\cos\varphi_2\}=0.\end{aligned}$$

Hence if two trajectories have been computed, the others can be found by these formulas. This gives a convenient process for sketching with great accuracy provisional trajectories for both air and field firing tables, with relatively little calculation. How best to select the two basic calculated trajectories is investigated in detail. *W. E. Milne.*

Sauer, R. und Pösch, H. Integriermaschine für gewöhnliche Differentialgleichungen. Z. Verein. Deutsch. Ingenieure **87**, 221–224 (1943). [MF 8504]

A brief description of a recently developed German machine of the differential analyzer type for solving ordinary differential equations. The machine uses wheel and disk integrators, drum and carriage input and output units, differential gear type adders and the elements are interconnected electrically with some sort of teletorque unit. Accuracy in solving

$$\frac{d^2x}{dt^2}+x=0$$

is given as .03 percent error in amplitude per cycle. Interconnection diagrams are shown for a number of simple equations. *C. E. Shannon* (New York, N. Y.).

Vietoris, L. Zur Theorie der Integraphen. Jber. Deutsch. Math. Verein. **52**, 71–74 (1942). [MF 8535]

The author makes a harmonic analysis of the difference between the true curve and the curve actually followed by the tracing point of the instrument and, after adopting suitable assumptions, concludes that the resulting error is "practically nil." *W. E. Milne* (Corvallis, Ore.).

Kormes, Mark. A note on the integration of linear second-order differential equations by means of punched cards. Rev. Sci. Instruments **14**, 118 (1943). [MF 8238]

In an article by L. Feinstein and M. Schwartzschild [same Rev. **12**, 405–408 (1941); these Rev. **3**, 156] there was outlined a method of numerical integration of linear ordinary second order differential equations by means of a "special" multiplying tabulator built by the International Business Machine Company. In the present note the authors outline a method of accomplishing the same result by the use of "standard" Remington Rand printing multiplying punch. The integration of this method is somewhat slower and accuracy is limited to factors of six digits; on the other hand, several advantages are claimed for the present method based upon the fact that there is a printed tape showing all the details of the computation and that there is less chance of mechanical failures. *H. Poritsky.*

Brown, S. Leroy and Wheeler, Lisle L. The use of a mechanical synthesizer to solve trigonometric and certain types of transcendental equations, and for the double summations involved in Patterson contours. J. Appl. Phys. **14**, 30–36 (1943).

Further applications of the harmonic synthesizer constructed in 1939 at the University of Texas. In earlier papers the authors have described the "Use of the mechanical multi-harmonograph for graphing of functions and for solution of pairs of nonlinear simultaneous equations" [Rev. Sci. Instruments **13**, 493–495 (1942); these Rev. **4**, 91] and a "Mechanical method for graphical solution of polynomials" [J. Franklin Inst. **231**, 223–243 (1941); these Rev. **2**, 240]. *R. L. Dietzold* (New York, N. Y.).

Härtel, W. Zur Theorie visuell beobachteter Oszillogramme von zeitaufgelösten periodischen Vorgängen. Z. Instrumentenkunde **63**, 132–140 (1943). [MF 8809]

Oblaski, Jan. Über einige mathematische Instrumente mit einer Messrolle, deren Achse mit Gewinde versehen ist. Z. Instrumentenkunde **63**, 100–108 (1943). [MF 8812]

Rauh, Kurt. Die Kurbelkurve des symmetrischen Doppelkurbelgetriebes mit dem Hub "0". Z. Instrumentenkunde **63**, 140–145 (1943). [MF 8810]

Korff, Günther. Ein Ausgleichsverfahren für die Koeffizientenbestimmung in der Potenzreihenentwicklung der sphärischen Aberration. Z. Instrumentenkunde **63**, 81–90 (1943). [MF 8811]

Hansel, C. W. An extension of nomography. Philos. Mag. (7) **34**, 1–26 (1943). [MF 8015]

In spite of the title, the paper consists only of examples of nomograms which are either well known or could easily be constructed by anyone familiar with the subject. Moreover, these nomograms are described without any discussion of general principles, and in terms of a special system of notation which obscures their actual simplicity. The paper has for its apparent purpose a polemic with imaginary antagonists who are "unaware of the existence" of, or "exhibit unreasoned hostility to," nomography. In keeping with the title, the section headings carry exaggerated and misleading implications as to section contents. "Numerical differentiation and integration" turns out to be the well-known nomogram for $y=ax^n$. The climax is nomogram 3, which is "capable of carrying out all general types of calculation." *P. W. Ketchum* (Urbana, Ill.).

Silver, R. S. and Weir, G. and J. The determination of turning values by means of logarithmic graphs. J. Sci. Instruments **20**, 76–77 (1943). [MF 8446]

Meincke, H. Annäherung der logarithmischen Spirale durch Kreisbogen. Z. Angew. Math. Mech. **22**, 168–169 (1942). [MF 8901]

Schminke, H. Eine Schieberanordnung für die Schlüsselgleichung $f_1(\varphi(\alpha)+\psi(\beta))+f_2(\gamma)=f_3(\alpha,\beta,\gamma)$, wobei nur das Endresultat abzulesen ist. Z. Angew. Math. Mech. **22**, 169–170 (1942). [MF 8902]

Babini, J. Graphical determination of real and complex roots of cubic equations. An. Soc. Ci. Argentina **134**, 309 (1942). (Spanish) [MF 8432]

Elkins, Thomas A. Nomograms for computing tidal gravity. Geophysics **8**, 134–145 (1943). [MF 8326]

Garvey, S. J. and Hetzel, K. W. Analytical geometry in common layouts. I. The case of retraction of an undercarriage about a single axis. Aircraft Engrg. **15**, 132–134, 143 (1943). [MF 8426]

The authors recommend the use of plane analytic geometry in planning the layouts for technical drawings and give an example in full detail. *E. Lukacs* (Jacksonville, Ill.).

RELATIVITY

Birkhoff, George D. Matter, electricity and gravitation in flat space-time. Proc. Nat. Acad. Sci. U. S. A. **29**, 231–239 (1943). [MF 8953]

[The author's New Theory of Relativity was first presented formally before the Astrophysical Congress in Puebla, Mexico on February 20, 1942, and will appear in the Proceedings of the Congress, cf. also Revista Ci., Lima **44**, 253–257 (1942); these Rev. **4**, 116.]

If one writes the equations of Einstein's theory of gravitation for g_{ik} differing but infinitely little from the constant values δ_{ik} characteristic for flat space-time, $g_{ik}=\delta_{ik}+h_{ik}$, and neglects higher powers of h_{ik}, one obtains a theory with a symmetric tensor of gravitation h_{ik} in a flat world which in its factual consequences agrees almost completely with Einstein's original theory. The ponderomotoric force of gravitation is a quadratic function of the velocity vector u_i. But of course the connection between metric and gravitation is dissolved and the proportionality of inertial mass and gravity has again become as mysterious as it was before Einstein. The author's set-up is essentially of this nature. He operates in the frame of the special theory of relativity and describes the state of the world by a scalar field ("matter"; Schrödinger's scalar wave equation), a vector (electromagnetic potential; Maxwell's equations), and a symmetric tensor h_{ik} (gravitational potential). His energy tensor forming the right member of the gravitational equations is that of a perfect fluid in which pressure and density are so related that small disturbances propagate with the velocity of light, $T_{ik}=2\rho u_i u_k+\rho\delta_{ik}$. The perfect fluid thus appears as a primitive irreducible physical entity. There seems no indication that the mechanical equations spring from a universal law of conservation of energy and momentum (which would comprise a gravitational part besides the "material" and electromagnetic parts).

H. Weyl (Princeton, N. J.).

Milne, E. A. Rational electrodynamics. III. The charge as point singularity. Philos. Mag. (7) **34**, 197–211 (1943). [MF 8256]

[Parts I and II appeared in Philos. Mag. (7) **34**, 73–82, 82–101 (1943); these Rev. **4**, 226, 227. Cf. also the following two reviews.] The author develops a new electrodynamics in the frame of kinematical relativity. The method used is deductive and based on identifications. The field is derived from singularities and the equations are not conditions imposed on the field, but identities satisfied by the field. The electromagnetic field $\mathbf{H}_1$, $\mathbf{E}_1$ defined at $(\mathbf{P}_1, t_1)$ by a test charge moving with the velocity $\mathbf{v}_1$ is

$$(\mathbf{H}_1)_s=\sum_{s=2}^{n}\left[\frac{\partial^2\Phi_{s1}}{\partial y_1\partial z_s}-\frac{\partial^2\Phi_{s1}}{\partial z_1\partial y_s}\right],$$

$$(\mathbf{E}_1)_s=\sum_{s=2}^{n}\left[\frac{\partial^2\Phi_{s1}}{\partial x_1\partial t_s}-\frac{\partial^2\Phi_{s1}}{\partial t_1\partial x_s}\right],$$

where the superpotentials Φ_{s1} are defined with the help of the source charges $e_s\, dt(\mathbf{P}_s, t_s)$ moving with velocities $\mathbf{v}_s$ by

$$\Phi_{s1}=\frac{\frac{1}{2}e_s}{y_1^{\frac{1}{2}}y_s^{\frac{1}{2}}}\cdot\frac{z_s z_{s1}}{(x_{1s}^2-x_1x_s)^{\frac{1}{2}}},$$

$$x_1=t_1^2-\mathbf{P}_1^2/c^2;\quad x_s=t_s^2-\mathbf{P}_s^2/c^2;\quad x_{1s}=t_1t_s-\mathbf{P}_1\mathbf{P}_s/c^2;$$

$$y_1=1-\mathbf{v}_1^2/c^2;\quad y_s=1-\mathbf{v}_s^2/c^2;$$

$$z_s=t_s-\mathbf{P}_s/c^2;\quad z_{s1}=t_s-\mathbf{P}_s\mathbf{v}_1/c^2.$$

The field equations satisfied by these expressions are

$$\operatorname{div}\mathbf{H}_1=0;\quad \operatorname{div}\mathbf{E}_1=(-1/c)\partial a_1/\partial t_1;$$

$$\operatorname{curl}\mathbf{E}_1=(-1/c)\partial\mathbf{H}_1/\partial t_1;\quad \operatorname{curl}\mathbf{H}_1=(1/c)\partial\mathbf{E}_1/\partial t_1+\partial a_1/\partial\mathbf{P}_1;$$

$$a_1=\sum_{s=2}^{n}\Box_1\Box_s\Phi_{s1},$$

which set of equations differs slightly from Maxwell's equations because $a_1\neq 0$. Moreover, the author deduces the equations of motion whose classical counterparts are Lorentz's equations of motion. *L. Infeld.*

Milne, E. A. Rational electrodynamics. IV. The "radius" of a point charge. Philos. Mag. (7) **34**, 235–245 (1943). [MF 8410]

The general theory of part III [cf. the preceding review] is now applied to the one body problem. The equations of motion and those of the field are used and then the t measure is replaced by the τ measure of time $(\tau=t_0\log t/t_0+\tau_0)$. The calculations are approximate. Two constants W and H appear in the process of integration; W represents energy only if $|e_1e_2|/mc^2$, called "the radius of the electron," is very small compared with the distance r between the particle with mass m, charge e_1 and the charge e_2 with infinite mass. Thus the concept of a length "the radius of the electron" appears automatically. Similarly H, the other constant, represents the angular momentum if r is much greater than the radius of the electron. To compare the results with the classical ones the author asks: what is the equivalent central force f which would give the same orbit in ordinary Newtonian nonrelativistic mechanics for an angular momentum equal to H? The answer is

$$f=\tfrac{1}{2}(e_1e_2/r^2)[(1+W/mc^2)^2e^{-e_1e_2/mc^2r}+e^{e_1e_2/mc^2r}].$$

This formula brings in apparent non-Coulomb forces at distances comparable to $|e_1e_2|/mc^2$. *L. Infeld.*

Milne, E. A. Rational electrodynamics. V. The neutron and nuclear dynamics. Philos. Mag. (7) **34**, 246–258 (1943). [MF 8411]

The case of a system composed of a proton $(M=\infty)$ and electron is investigated with the help of quantum mechanics. That is, in the formulae of part IV [cf. the preceding review] the constant H is assumed to be $H=nh/2\pi$. Thus, as the author says: "we are departing from our present strictly deductive scheme, and making a special assumption, a procedure entirely foreign to our proper purpose." What the author does corresponds to Bohr's old theory of 1913 with the difference that the classical background is that of Milne's and not of Maxwell's electrodynamics. Accordingly, the results are different. In addition to systems with Bohr's orbits, there exist systems with circular orbits with radii approximating $\frac{1}{2}e^2mc^2$ which are identified as neutrons. The radius of the neutron changes little with the quantum number n. Although these circular orbits are unstable an electron will, in general, spend a long time in their vicinity before either escaping to infinity or falling into the nucleus. It follows further that the proton appears to possess a potential barrier at the distance $\frac{1}{2}e^2/mc^2$; this is due to the new dynamics and not to the breakdown of Coulomb's law. *L. Infeld.*

Whitrow, G. J. Axiomatic treatment of kinematical relativity: a reply to Dr. G. C. McVittie. Proc. Roy. Soc. Edinburgh. Sect. A. 61, 298–299 (1943). [MF 8299]

Milne, E. A. On the equation of motion of a free particle in the expanding universe of kinematical relativity. Proc. Roy. Soc. Edinburgh. Sect. A. 61, 288–297 (1943). [MF 8300]

Both papers are of a polemic character and are directed against McVittie [Proc. Roy. Soc. Edinburgh. Sect. A. **61**, 210–222 (1942); these Rev. **4**, 174] who claims he has derived Milne's formula for the acceleration of a free particle moving in the presence of a substratum. Milne claims that McVittie deduced this result from the axioms of kinematical relativity "together with an additional assumption, which is equivalent to begging the answer to the whole problem." *L. Infeld* (Toronto, Ont.).

MacColl, L. A. The fundamental equations of electron motion. Dynamics of high speed particles. Bell System Tech. J. **22**, 153–177 (1943). [MF 8694]

Starting with the well-known equations

$$\frac{d}{dt}\left[\frac{m_0\dot{x}_n}{(1-(v^2/c^2))^{\frac{1}{2}}}\right]=X_n, \qquad n=1, 2, 3,$$

for relativistic particle dynamics, the author develops for purely expository purposes the corresponding Lagrangian and Hamiltonian equations, the appropriate variational principles and the Hamilton-Jacobi theory. *D. C. Lewis.*

Würschmidt, Jose. Shock of variable masses and the law of reflexion of a particle. Univ. Nac. Tucumán. Revista A. **3**, 79–110 (1942). (Spanish) [MF 8155]

The paper consists of a detailed study of the impact of two relativistic masses based on the theorems of conservation of energy and of momentum. As an application the known law of refraction of the light on a moving mirror is obtained. *I. Opatowski* (Chicago, Ill.).

Beck, Guido. Sur la possibilité d'une cinématique générale. Centro de Estudos de Mat. Fac. Ci. Pôrto. Publ. no. **5**, 12 pp. (1943). [MF 8783]

The usual interpretation of the Dirac wave equation for an electron is based upon the idea that the electron is a particle, having certain special characteristics such as charge and spin, in the ordinary space-time of special relativity. The equation is regarded as a dynamical relation governing the motion of the particle. The author suggests an alternative, and he believes preferable, interpretation. According to this new interpretation the significance of the Dirac equation is essentially geometrical, in that the equation concerns the structure of an "electronic space-time." The laws of the electromagnetic field are obtained as consequences of the microstructure of this electronic space-time, somewhat as the laws of the gravitational field are obtained as consequences of the macrostructure of space-time in general relativity. The note is too brief to give an adequate exposition of the new ideas. *L. A. MacColl.*

MATHEMATICAL PHYSICS

Wendt, Georg. Zur Dioptrik elektronenoptischer Geräte mit beliebig gekrümmten Abbildungsachsen. Z. Phys. **120**, 720–740 (1943). [MF 8492]

Starting directly from Fermat's law, first order laws (Gaussian optics) along a general curved ray in an electro-optic medium are derived by determining potentials and electric and magnetic field vectors in curvilinear coordinates. The author gives formulae for the stigmatic image formation within the accuracy of his approximation and discusses the purely magnetic field and the purely electric field. *M. Herzberger* (Rochester, N. Y.).

Duntley, Seibert Q. The mathematics of turbid media. J. Opt. Soc. Amer. **33**, 252–257 (1943). [MF 8337]

This paper is a short summary of the main lines of investigation into the propagation of light in turbid media. After a few remarks on primary scattering and simple absorption, the author discusses the theories of secondary scattering developed by Wiener, Schuster and King. Wiener's method was statistical, while Schuster set up differential equations for the light flux moving in two opposite directions through the medium; but neither method affords a general solution of the problem. The desired general solution was given in the form of an integral equation by King in 1913, and the author reviews the derivation of this equation, pointing out that useful solutions, applicable to industrial turbid materials, have not yet been produced, and recommending the subject to the attention of mathematicians. Finally he presents an example of the application of the method of Schuster to an engineering problem. The paper is valuable in drawing attention to the possibilities latent in the elegant method of King. *W. E. K. Middleton* (Toronto, Ont.).

Sivukhin, D. V. Contribution to the molecular theory of light reflection. C. R. (Doklady) Acad. Sci. URSS (N.S.) **36**, 231–234 (1942). [MF 8092]

The problem of reflection is reduced by a simple device to the solution found by Ewald for the field excited by a dipole wave of constant amplitude. Use then is made of a theorem of incident wave damping, quantities of order $(a/\lambda)^2$ are neglected (a being the lattice constant and λ the wave-length) and it is found that to this approximation a certain vector P can be found by solving an electrostatic problem. The formulae for the amplitudes in the reflected waves are said to be valid also for a lattice on whose surface there is a layer of transition whose thickness is small compared with λ and whose constituent atoms in the parallel layers differ from those of the lattice. Lattices with continuous transition layers and lattices without transition layers are considered. Finally an attempt to apply the theory to a liquid is made and the role of transition layers is seen to be important. *H. Bateman.*

Schriever, O. Eine anschauliche Darstellung der Theorie der inhomogenen ebenen Welle. Hochfrequenztech. Elektroak. **60**, 100–104 (1942). [MF 8514]

Elementary discussion of reflection and refraction of non-homogeneous plane waves at the interface between two dissipative media. Instead of "complex" angles the author considers separately the angles of incidence, reflection and refraction for the planes of constant phase and for those of constant amplitude. Total reflection and the reflection and refraction coefficients for the field vectors are also discussed. *M. C. Gray* (New York, N. Y.).

Ginsburg, V. On the reflection of an electromagnetic impulse from the Heaviside-layer. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **12**, 449–459 (1942). (Russian) [MF 8407]

The author makes a theoretical analysis, in the framework of geometric optics, of the reflection of a square-pulse-modulated radio wave of carrier-frequency ω_0 from the Heaviside layer. The frequency spectrum (Fourier transform) of this impulse is obtained and then subjected to the condition for reflection, using the concept of the phase-integral. Both square-law and linear electron-density distributions in the Heaviside layer are treated (and qualitative discussion given for other distributions). These lead to Fresnel integrals and from them conclusions are drawn as to the time required for the establishment of the main body of the reflected impulse. The author states that in certain practical cases he obtains answers, without fudging, in good agreement with experiment. *H. Wallman.*

Försterling, K. Über die Ausbreitung elektromagnetischer Wellen in einem magnetisierten Medium bei senkrechter Inzidenz. Hochfrequenztech. Elektroak. **59**, 10–22 (1942). [MF 8510]

It is shown that a plane electromagnetic wave propagating in an ionized medium under the influence of a constant magnetic field can split only into two plane waves which are elliptically polarized in opposite directions. This result was originally deduced from the theory of magneto-optics [K. Försterling and H. Lassen, Ann. Physik **18**, 26 (1933)]. In the present paper it is assumed that propagation takes place in the z direction, that electron density and therefore all field vectors are functions only of z, and that the angle between the constant magnetic field and the z direction is α. This leads to the differential equation

$$\left\{\epsilon^{(1)}(z)-\frac{1}{(1+q^2)^2}\left(\frac{\partial q}{\partial z}\right)^2\right\}u^{(1)} = \frac{\partial^2 u^{(1)}}{\partial z^2}+i\left\{u^{(2)}\frac{\partial}{\partial z}\frac{\partial q/\partial z}{1+q^2}+\frac{\partial u^{(2)}}{\partial z}\cdot 2\frac{\partial q/\partial z}{1+q^2}\right\}$$

and another equation with superscripts interchanged. Here $\epsilon^{(1)}(z)=D_x^{(1)}/E_x^{(1)}$ is an equivalent dielectric constant of the wave of the first type and a complicated function of z and so is q. Furthermore, $u^{(1)}=(1+q^2)^{\frac{1}{2}}E_x^{(1)}$ and $u^{(2)}=(1+q^2)^{\frac{1}{2}}E_y^{(2)}$, so that the bracketed term on the right-hand side represents a coupling term between the two waves, which vanishes if q does not depend on z, that is, if the electron density remains constant. In that case, the differential equation obviously reduces to the conventional wave equation. The z-dependent terms render an exact solution impossible so that the author assumes at first negligible coupling terms, finds the fundamental solutions $v^{(1)}$ and $v^{(2)}$ for the simplified differential equations and then uses these solutions as known functions on the right-hand sides. This gives two inhomogeneous second order differential equations which can be solved. The simplified fundamental solutions $v^{(1)}$ and $v^{(2)}$ each contain two terms and the main part of the paper is devoted to a detailed discussion of these terms under various practical assumptions of electron density variation, total reflection, small double reflection and absorption. Under no condition will the two terms of either $v^{(1)}$ or $v^{(2)}$ constitute two separately observable waves. It appears that this will be true even if higher order approximations are considered. Since experimentally occasional extra waves have been observed, a basic modification of the theory seems warranted. *E. Weber.*

Goubau, Georg. Reziprozität der Wellenausbreitung durch magnetisch doppelbrechende Medien. Hochfrequenztech. Elektroak. **60**, 155–160 (1942). [MF 8747]

Assuming nonisotropic media so that permeability, dielectric constant and conductivity become tensors, relations are deduced from Maxwell's field equations which correspond to the reciprocity theorem of Lorentz and reduce to it for isotropic media. Transforming the relations into four pole terminology, applications are made to antenna systems and it is shown that reciprocity for wave propagation in the ionosphere depends upon the type of radiator used.

E. Weber (Brooklyn, N. Y.).

Grünberg, G. A. Theory of the coastal refraction of electromagnetic waves. Acad. Sci. USSR. J. Phys. **6**, 185–209 (1942). [MF 8371]

The coastal refraction of electromagnetic waves is treated by assuming that the land surface is practically level with the sea. If the effect of conductivity of the soil exceeds sufficiently the effect of displacement currents, the boundary condition

$$\partial E_z/\partial z=\text{const. } E_z$$

is shown to hold approximately over the land surface $z=0$; this condition is obtained by neglecting the x and y derivatives in the wave equation in comparison with the z-derivative. By means of it and the assumption that over the surface of the sea the field is normal to it, the problem is reduced to that of a single medium with proper boundary conditions. By applying Green's double layer theorem and utilizing these conditions, an integral equation is obtained for the values of E_z over $z=0$. Approximate solutions of this equation are indicated and, in particular, the case of an infinitely long rectilinear coast $x>0$ with the field varying sinusoidally in y is investigated. *H. Poritsky.*

Sommerfeld, A. und Renner, F. Strahlungsenergie und Erdabsorption bei Dipolantennen. Hochfrequenztech. Elektroak. **59**, 168–173 (1942). [MF 8505]

Starting from the known formulas for the electromagnetic field of a vertical or horizontal dipole at height h above a plane homogeneous earth the authors determine the radiated power by integrating the complex Poynting vector over two parallel planes at heights $h\pm\epsilon$. This leads to known formulas for a radiation characteristic s for either dipole as a function of h. For a perfectly conducting earth, as $h\rightarrow\infty$, $s_v=s_h=\frac{2}{3}$, the free space value, while as $h\rightarrow 0$, $s_v=\frac{4}{3}$ and $s_h=0$. If the earth is not perfectly conducting a correction term is obtained, and it is shown that as the dipole approaches the earth's surface this term becomes infinite like h^{-3}. While this is mathematically true for an ideal dipole it should be pointed out that the authors are not justified in assuming that their formulas can be applied to determine the radiation resistance of finite antennas unless the earth is a perfect conductor or unless finite height values are retained. For a finite antenna immediately above an imperfectly conducting earth the finite transverse dimensions must be taken into account. *M. C. Gray.*

Grosskopf, J. Das Strahlungsfeld eines vertikalen Dipolsenders über geschichtetem Boden. Hochfrequenztech. Elektroak. **60**, 136–141 (1942). [MF 8507]

The author obtains a practical solution of the problem of the vertical electric dipole above a stratified earth by determining a reflection coefficient corresponding to that for homogeneous earth. The resulting field formulas have

the same form as in Weyl's solution for the homogeneous case when, in the Sommerfeld numerical distance definition, the refractive index (or complex dielectric constant) of the top layer is replaced by an effective value which depends on the properties and thicknesses of the successive layers.

M. C. Gray (New York, N. Y.).

Grosskopf, J. Über das Zenneckscke Drehfeld im Bodenwellenfeld eines Senders. Hochfrequenztech. Elektroak. **59**, 72–78 (1942). [MF 8508]

In 1907 Zenneck discussed the propagation of plane waves over a finitely conducting plane earth and found that the resulting electric field was elliptically polarized in the plane of propagation. The author of the present paper shows that this polarized field can be obtained as the resultant of a direct and reflected wave, the latter being determined by using the Fresnel reflection coefficient. For near grazing incidence this method leads to an approximate formula

$$E_z/E_y = 1/\sqrt{\epsilon'},$$

where ϵ' is the complex dielectric constant, E_z the field component tangential to the surface and E_y the normal component. If the incident wave is nonhomogeneous (that is, planes of constant amplitude no longer coincident with planes of constant phase), a similar formula can be obtained by a suitable definition of a complex angle of incidence. The author also discusses the exact solution of the problem of a vertical dipole above a plane earth and shows that the Zenneck field is a valid approximation to this solution within certain ranges of the height of the dipole and of the distance at which the field is measured. *M. C. Gray.*

Harrison, Charles W., Jr. Radiation from vee antennas. Proc. I.R.E. **31**, 362–364 (1943). [MF 8819]

Formulas for the directivity of an inclined vee antenna are derived under the simplifying assumptions of harmonic current distribution and perfectly conducting earth. Nonresonant operation is assumed. The favorable directional characteristics over a wide band and other advantages of this type of aerial are pointed out. *R. M. Foster.*

Borgnis, F. Die elektrische Grundschwingung des kreiszylindrischen Zweischichten-Hohlraums. Hochfrequenztech. Elektroak. **59**, 22–26 (1942). [MF 8511]

The axially symmetrical solution of Maxwell's field equations for the E-modes of the circular cylindrical resonator is extended to the case of two concentric dielectric layers of different dielectric constants. Assuming as radial boundary conditions the vanishing of the axial electric field on the surface of the resonator and continuity of the same quantity on the boundary surface of the two dielectrics leads to the "characteristic equation" for the resonant frequencies involving Bessel functions of the first and second kinds. With restriction to the lowest mode (lowest resonant frequency), the graphically obtained solution is discussed as a function of the radius of the inner rod and the ratio of the two dielectric constants, and several graphs of radial field distributions are given. Approximations are derived for very thin rods as a basis for experimental determination of dielectric constants at very high frequencies.

E. Weber (Brooklyn, N. Y.).

Borgnis, F. Die magnetische Grundschwingung des kreiszylindrischen Hohlraums. Hochfrequenztech. Elektroak. **60**, 151–155 (1942). [MF 8746]

In cylindrical electromagnetic resonators the lowest mode (longest resonant wave length) is associated with the dominant magnetic mode existing in circular wave guides as long as the axial length $l > 2.02R$, if R is the radius of the cylinder. The author investigates the dependence of the resonant wave length on the axial length and computes the Joule losses which show a definite minimum at $R/l = 0.66$.

E. Weber (Brooklyn, N. Y.).

Meinke, H. Die Eigenwellen des belasteten zylindrischen Hohlraums. Hochfrequenztech. Elektroak. **60**, 29–37 (1942). [MF 8512]

In order to approximate the resonant frequencies of circular cylindrical electromagnetic resonators with electrical excitation or load along an inner coaxially cylindrical surface, the basic solution of Maxwell's field equations is reinterpreted in terms of transmission line equations with cylindrical waves traveling radially out and being completely reflected at the outer cylindrical boundary surface, which is equivalent to an electrical short circuit. All discussions are restricted to axially short resonators (short compared with wave length and outer radius) so that only the lowest E-mode need be considered. Expressions are derived for the equivalent load impedance to be placed at the center of the resonator under different conditions. Experimental results check the approximate theory very well in the cases of excitation through a gap, loading by a coaxial line in the center and loading by a dielectric rod in the center. The approximate theory is also extended to eccentric application of loads. *E. Weber* (Brooklyn, N. Y.).

Middleton, David. Ultra-high frequency oscillations of cylindrical cavity resonators containing two and three dielectric media. Phys. Rev. (2) **63**, 343–351 (1943). [MF 8339]

The author first obtains a formal solution in terms of the Hertzian vector potentials for the problem of a perfectly conducting cylindrical cavity resonator containing three distinct dielectric media. Resonance conditions for the different modes of oscillation which may be excited in the resonator are outlined, and detailed solutions are given for a movable slab of dielectric in a resonator of fixed length and in one of variable length. Experimental measurements of the resonant lengths in these cases were found to agree closely with the theoretical values. Finally it is shown that a variable resonator with a dielectric slab at one end may be used for the measurement of dielectric constants at ultra-high frequencies. *M. C. Gray* (New York, N. Y.).

Pipes, Louis A. Electrical circuit analysis of torsional oscillations. J. Appl. Phys. **14**, 352–362 (1943).

One of the two well-known analogies between linear mechanical systems and electrical circuits is applied to torsional systems, letting the current correspond to the angle of torsion (not, as usual, to the angular velocity). Lumped systems, in particular, multi-cylinder engines with short shafts, as well as systems with long (continuous) shafts are treated, without and with consideration of viscous damping. The analytical expressions for the natural frequencies and decrements are derived for the simplest recurrent structures, otherwise the experimental method of obtaining them on the bench with oscillator and ammeter is pointed out. This includes forced oscillations. In case the torsional system includes gears, the equivalent circuits contain transformers.

H. G. Baerwald (Cleveland, Ohio).

Maa, Dah-You. A general reactance theorem for electrical, mechanical, and acoustical systems. Proc. I.R.E. 31, 365–371 (1943). [MF 8820]

Three "extensions" of the well-known Foster-Cauer reactance theorem are treated, namely, from electrical to mechanical and acoustical lumped systems with single drive, to distributed finite systems and to systems with multiple (coherent) driving forces. None of these "extensions" seems new nor deserves the name. The first is literally equivalent to the electrical theorem and precedes it historically (Routh). No proof is indicated for the second statement, which is also well known; several examples such as the vibrating string and bar are given. The third extension is merely a reformulation of the original theorem; the "extended" reactance (impedance) concept has long been used in the treatment of acoustical and electro-acoustic systems, various quotations being given by author.

H. G. Baerwald (Cleveland, Ohio).

West, S. S. The mutual impedance of collinear grounded wires. Geophysics 8, 157–164 (1943). [MF 8327]

The formula due to R. M. Foster [Bell System Tech. J. 10, 408–419 (1931)] for the mutual impedance of grounded wires lying on the surface of a homogeneous earth is evaluated for the case of straight wires grounded at equi-spaced collinear points. If the four grounding points are numbered consecutively, electrodes 1 and 2 are connected to form one circuit, 3 and 4 to form the other. Curves are given to show the absolute value and the angle of the mutual impedance as functions of resistivity and frequency, with a uniform separation between adjacent electrodes of 1000 feet. These results are tabulated for the general case by taking as the independent variable the product of the electrode separation and the square root of the ratio of frequency to resistivity, as the dependent variable the product of the electrode separation and the complex impedance divided by the resistivity, with appropriate numerical factors. The sign of the mutual impedance seems to have been reversed in the course of the manipulation of formulae in this paper; also the factor 10^6 should be omitted from the heading of the second column in Table I on page 163. *R. M. Foster.*

Wiechowski, W. Zur Stromverdrängung beim koaxialen Kabel. Hochfrequenztech. Elektroak. 59, 174–179 (1942). [MF 8506]

In the conventional theory of wave propagation in coaxial cables, the propagation constant and the characteristic impedance are derived from values of self-inductance and resistance per unit length, which are computed by the conventional skin-effect theory, that is, under the assumption that the field does not depend on the direction of propagation. While this inconsistency does not cause appreciable errors as long as the cross-dimensions of the cable are small compared with the wavelength, it is questionable whether the method will stand up in case of modern wide-band coaxials. The author applies rigorous field theory and obtains the determinantal equation for the propagation constant for the general case of three concentric cylindrical media. The result is specialized to the case that the core and sheath material is a conductor, and the intermediate one an insulator with small loss angle. It is shown that, in the limit, the result goes over into that of the conventional theory. *H. G. Baerwald* (Cleveland, Ohio).

Poritsky, H. Field concentration near circular conductors. Bull. Amer. Math. Soc. 49, 417–421 (1943). [MF 8390]

Approximate computation of the field concentration factor in an electric cable consisting of a row of circular wires tightly arranged about a circular core and surrounded by circular sheath concentric with the latter. If the ratios: radii of external-to-core wires and of core-to-sheath are small compared with one, the configuration may be approximated by a linear row of mutually tangent semicircles opposing at great distance a parallel line which simulates the sheath. By means of the modular function which connects modulus and period ratio of elliptic functions, the strip domain bounded by a quarter circle and extending to infinity is mapped on a quarter plane. The field concentration factor is obtained from the modulus of the derivative of this function at the circle crest via rapidly converging ϑ-series. *H. G. Baerwald* (Cleveland, Ohio).

König, H. Die Ähnlichkeitsgesetze des elektromagnetischen Feldes und ihre Anwendung auf Elektronenröhren. Hochfrequenztech. Elektroak. 60, 50–54 (1942). [MF 8513]

Electromagnetic fields are studied from the point of view of the theory of similarity or dimensional analysis. Questions such as under what conditions will the change of size yield the same ratios between convection and displacement currents are discussed. The results are applied to electron tubes such as triodes, klystrons and magnetrons.

H. Poritsky (Schenectady, N. Y.).

Päsler, Max. Die Anwendung des Matrizenkalküls auf Probleme der HF-Technik. Hochfrequenztech. Elektroak. 59, 78–86 (1942). [MF 8509]

An expository paper in which the author first describes some elementary properties of matrices and then uses these properties to solve the simple problems of coupled circuits and of transformers connected in series. *M. C. Gray.*

Picht, Johannes und Himpan, Josef. Beiträge zur Theorie der elektrischen Ablenkung von Elektronenstrahlenbündeln. I. Allgemeine Untersuchungen über den Strahlenverlauf in elektrostatischen Ablenkfeldern. Ann. Physik (5) 39, 409–435 (1941). [MF 8729]

Picht, Johannes und Himpan, Josef. Beiträge zur Theorie der elektrischen Ablenkung von Elektronenstrahlenbündeln. II. Elektrische Ablenkung eines (ausgedehnten) elektronenoptischen Bildes und die dabei auftretenden Bild- und Ablenkfehler bis zur dritten Ordnung. Ann. Physik (5) 39, 436–477 (1941). [MF 8730]

Picht, Johannes und Himpan, Josef. Beiträge zur Theorie der elektrischen Ablenkung von Elektronenstrahlenbündeln. III. Dynamischer Bildaufbau mittels gekreuzter elektrischer Ablenksysteme und die dabei auftretenden Abbildungsfehler bis zur dritten Ordnung. Ann. Physik (5) 39, 478–501 (1941). [MF 8731]

Die nachfolgenden drei Beiträge gehören der Untersuchungsmethode sowie dem Inhalt nach eng zusammen, obwohl sich besonders die Beiträge II and III auf verschiedene Anwendungsgebiete beziehen. Im Beitrag I, werden die für II und III benötigten allgemeinen Grundformeln für die Ablenkung von Elektronenstrahlen durch elektrische Ablenksysteme abgeleitet. Der Beitrag II wendet diese Formeln auf die Ablenkung eines ausgedehnten elektronenoptischen Bildes an und diskutiert die hierbei auftre-

tenden Bild- und Ablenkfehler ausführlich, wobei zunächst nur die Ablenkung mittels eines elektrisch geladenen Plattenpaares konstanter Ablenkspannung—als "statische Ablenkanordnung" bezeichnet—behandelt wird. Diese Untersuchungen sind z. B. von Wichtigkeit für die theoretische Behandlung der Vorgänge in bestimmten Bilderzerlegerröhren, für gewisse Fragen der Massenspektroskopie, für Mehrfachoszillographen usw. Der Beitrag III behandelt unter teilweiser Benutzung der Ergebnisse der beiden vorhergehenden Beiträge den für Fernsehzwecke wichtigen (dynamischen) Bildaufbau mittels zweier gekreuzter Ablenksysteme variabler Spannung ("dynamische Ablenkanordnung") und diskutiert eingehend die hierbei auftretenden Bildfehler. *Author's summary.*

v. Laue, M. Bemerkungen zur Supraleitung. Phys. Z. **43**, 274–284 (1942). [MF 8621]

The author discusses some difficulties in the interpretation of superconductivity, namely, the absence of a noticeable contribution to the energy of order depending on the current, the apparent lack of mobility of the current lines as exhibited by hollow spheres and superconducting rings, etc. A discussion of the experiments on magnetic threshold values is given on the basis of the equilibrium condition formerly developed by the author [Ann. Physik (5) **32**, 71–84 (1938)] for the equilibrium at the boundary between the superconducting and the nonsuperconducting phases. This condition is known to lead to inconsistencies unless further assumptions regarding a surface tension on the boundary between the two phases are made [H. London, Proc. Roy. Soc. London. Ser. A. **152**, 650–663 (1935)]. In fact the author's formula for the magnetic threshold value as a function of the film thickness derived on the basis of this incomplete equilibrium condition is not in agreement with the measurements of Appleyard and others.

F. W. London (Durham, N. C.).

v. Laue, M. Nochmals über Stromverteilung in Supraleitern. Z. Phys. **120**, 578–587 (1943). [MF 8490]

The minimum theorem of the current distribution in ramified superconductors [M. v. Laue, Z. Phys. **118**, 455 (1941)] is extended to systems consisting partially of normal conductors and partially of superconductors.

F. W. London (Durham, N. C.).

Laval, Jean. Composition des rayons X diffusés par un cristal perturbé par l'agitation thermique. C. R. Acad. Sci. Paris **214**, 274–276 (1942). [MF 7848]

The diffuse radiation is represented quite simply in the space of the reciprocal lattice. There is a fundamental radiation having the frequency of the incident X-rays and in addition waves produced by the thermal agitation. These are expressed by means of series whose coefficients are ratios of Bessel functions. The frequencies are of type

$$n+pn_\alpha+qn_\beta+\cdots,$$

where n is the frequency of the fundamental and p, q, $\cdots$ are integers. If the crystal is composed of N atoms, the radiations of the first order are $6N$ in number, those of the second order are $18N^2$ in number, $6N$ arising from one elastic wave and the rest from two elastic waves. The radiations of the third order take three different forms according as they are diffused by one, two or three of the elastic waves and so on. *H. Bateman* (Pasadena, Calif.).

Laval, Jean. La diffusion cristalline des rayons X peut être envisagée comme résultant de réflexions de Bragg, avec changement de fréquence, sur les plans d'ondes d'agitation thermique. C. R. Acad. Sci. Paris **214**, 431–433 (1942). [MF 7849]

An elastic wave which is propagated in a crystal is formed by the oscillation of the atoms which are analyzed into plane waves whose vectors of propagation and intervals are simply represented. The end of a vector of propagation drawn from the origin is called a pole of diffusion. When this does not coincide with the end of one of the fundamental vectors of propagation, a shift to the nearest of these ends will give a diffusing power which does not differ appreciably from the actual diffusing power. If the total number of atoms is N, each vector of propagation belongs to $3N$ elastic vibrations and the radiations selectively reflected are $6N$ in number. The intensity is proportional to the square of a series whose coefficients are ratios of Bessel functions. *H. Bateman* (Pasadena, Calif.).

Bond, Walter L. The mathematics of the physical properties of crystals. Bell System Tech. J. **22**, 1–72 (1943). [MF 8066]

A simple and concise treatment of the tensorial properties of crystals is given by means of matrix algebra. The methods of the latter are developed in the paper. After a preliminary discussion of symmetry classes and of space rotations, the number of tensor constants and the behavior under rotations are derived and collected for all crystallographic classes for the following effects: dielectric properties, elasticity, that is, the relations between stress and strain including their temperature coefficients, thermal expansion, the piezo-electric effect and its converse, pyro-electric and thermo-electric effects, the propagation of light and piezo-optic effects. *L. W. Nordheim* (Durham, N. C.).

Haenzel, G. Die Polygonfläche und das periodische System der Elemente. Z. Phys. **120**, 283–300 (1943). [MF 8325]

The author constructs a special figure by using regular polygons and circles in an elementary way. Certain points are singled out in this figure, given coordinates in a simple way and then interpreted as representing (1) the solutions of the Schrödinger equation for a one electron system, (2) the valence electrons of the chemical elements, (3) the protons of the nuclei of the elements. *A. Schwartz.*

Gora, E. Quantentheorie der Strahlungsdämpfung. Z. Phys. **120**, 121–147 (1943). [MF 8245]

The perturbation method used in the quantum theory of radiation is reformulated in a way which permits the systematic inclusion of longer chains of intermediate states leading to the same final state. Considerations of these terms, as far as they do not lead to divergences, lead to the inclusion of radiative reaction as in the equivalent independent treatments by W. Heitler [Proc. Cambridge Philos. Soc. **37**, 291–300 (1941); these Rev. **4**, 95] and A. H. Wilson [ibid. **37**, 301–316 (1941); these Rev. **4**, 95] and to a close correspondence to the classical formulas of this problem. The theory is applied to the scattering of light for a charged particle with spins 0 and 1. In the latter case, radiative reaction causes a considerable reduction of the scattering cross section at higher energies, which, in turn, gives rise to a reduction of the cross section for radiation on passage through matter at high energies. The scattering of

vector mesons by the nuclear field of nuclear particles is also treated, where a reduction of the scattering cross section occurs already at fairly low energies as required by cosmic ray evidence. *L. W. Nordheim* (Durham, N. C.).

Bhabha, H. J. and Chakrabarty, S. K. The cascade theory with collision loss. Proc. Roy. Soc. London. Ser. A. **181**, 267–303 (1943). [MF 8479]

The statement of physical assumptions relating to the chance of emission of a quantum of energy by an electron of energy E introduces a characteristic length l while the "ionization" or collision loss introduces a constant quantity β having the dimensions of energy. If $P(E,t)dE$ denotes the number of particles (electrons and positrons) whose energies lie between E and $E+dE$ at a distance t from the surface of the substance and if $Q(E,t)dE$ is the corresponding expression for the number of quanta, differential-difference equations are set up for two related quantities

$$p(s,t)=\int_0^\infty E^{s-1}P(E,t)dE,\quad q(s,t)=\int_0^\infty E^{s-1}Q(E,t)dE.$$

If d denotes the operator $\partial/\partial t$, these equations are

$$dp(s,t)+A_sp(s,t)-B_sq(s,t)+\beta(s-1)p(s-1,t)=0,$$
$$dq(s,t)-C_sp(s,t)+Dq(s,t)=0,$$

where A_s, B_s and C_s are functions of s whose values are found by integration. Elimination of q leads to the equation

$$d^2p(s,t)+(A_s+D)dp(s,t)+(A_sD-B_sC_s)p(s,t)+\beta(s-1)(d+D)p(s-1,t)=0.$$

The case of no collision loss ($\beta=0$) is first worked out in detail and the results are tabulated to 3 or 4 decimals for $s=1(.1)6$ so that the solution can be completed with the aid of the Mellin formula of inversion for the definite integrals representing p and q. An approximation is made by means of the saddle-point method and its accuracy is checked by some numerical work. The case $\beta\neq 0$ is next considered by more complicated analysis in which a function $g(s,t)$ is tabulated first by means of an approximate equation for $s=1.1(.1)5$ and then for comparison by means of an accurate equation for the set of values $s=1.5(.5)5$. The energy spectrum of electrons at different thicknesses is given in a table. *H. Bateman* (Pasadena, Calif.).

Schumann, W. O. Über Plasmalaufzeitschwingungen. Z. Phys. **121**, 7–33 (1943). [MF 8743]

The plasma is defined as the mixture of a stationary heavy positive ion gas through which a highly mobile electron gas is diffusing. The response of this plasma to externally impressed oscillations of either field strength, velocity or current density is investigated in great detail for many different boundary conditions. In particular, the possibilities of self excitation are studied. *E. Weber.*

Lifschitz, E. M. On the theory of phase transitions of the second order. II. Phase transitions of the second order in alloys. Acad. Sci. USSR. J. Phys. **6**, 251–263 (1942). [MF 8348]

[The first part appeared in the same J. **6**, 61–74 (1942); these Rev. **4**, 206.]

Further developing the theory of phase transitions of the second order (the study of Curie points) which he presented in an earlier paper, the author makes a full examination of the possible existence of phase transitions of the second order for order- and disorder-transitions in alloys. Solid solutions with body- or face-centered cubic lattices or with a lattice of the hexagonal close-packed type are investigated. The first step is to study the space group of the solid superstructure and find a set of functions ϕ_i which give rise to its irreducible representations. For instance, the functions $\cos\pi x\cos\pi y\cos z$ and $\sin\pi x\sin\pi y\sin\pi z$ are unchanged either by translation from the origin to any of the sites (0, 1, 1), (1, 0, 1), (1, 1, 0), (2, 0, 0), (0, 2, 0), (0, 0, 2), etc., of a face-centered cubic Bravais lattice, or by rotations of the point group associated with the elementary cell; but these functions change sign under a translation to the point (1, 1, 1). The terms in the expansion of the thermodynamic potential are obtained from invariants formed from these functions, and certain coefficients are then chosen so as to insure the existence of a minimum. Nonequivalent sites in the cell of the Bravais lattice are assigned to different atoms so as to form the ordered alloy. After a thorough discussion of the possible superstructures in the three types of Bravais lattice under investigation, the results are summarized in a table; the three superstructures for the body-centered lattice are shown in a figure. Finally the author discusses the relationship between his theoretical results and the rather scanty experimental data covering the phase transitions of the second order in alloys, characterized by the absence of latent heat and the presence of a discontinuity in the heat capacity. *J. S. Frame.*

Höcker, K. H. Wirkungsquerschnitte der Reaktionen zwischen Neutronen und Deuteronen. Phys. Z. **43**, 236–257 (1942). [MF 8622]

This paper deals with the calculation of the cross sections for scattering and absorption of neutrons by deuterons and disintegration of deuterons by neutrons. Exchange forces are taken into account by taking the interaction potential between any two particles as a linear combination of the Majorana, Heisenberg, Bartlett and Wigner forces with the radial dependence given by the Gauss error function and the various constants determined from the known properties of the deuteron. It is assumed that the wave function of the initial state can be written as the product of the functions $g(r)$, $f(R)$, where r is the distance between the two particles of the deuteron and R is the distance of the third particle from the center of gravity of the deuteron. This assumption corresponds to the neglect of the polarization of the deuteron. For $g(r)$ is taken the ground state eigenfunction of the deuteron, and for $f(R)$ for small values of R is assumed the function

$$(1+\beta_2R^2+\beta_4R^4)e^{-\eta R^2}$$

and the constants β_2, β_4 and η are determined by the solution of the Schrödinger equation. The phase of the incident neutron beam for large R is calculated from the condition that the above function join on smoothly with the asymptotic function for $f(R)$. The results of the calculations agree fairly well with observed values of the cross sections.
S. Kusaka (Northampton, Mass.).

Nikolsky, K. On the theory of mesons. C. R. (Doklady) Acad. Sci. URSS (N.S.) **38**, 173–175 (1943). [MF 8684]

A wave equation for the nucleon in the presence of an external meson field is derived by making an analogy with the Dirac equation for the electron in the presence of an external electromagnetic field. It is stated that this equation is invariant under all rotations in a five-dimensional Euclidean space, and the effect of this transformation on the wave function is investigated. *S. Kusaka.*

Heisenberg, W. Die "beobachtbaren Grössen" in der Theorie der Elementarteilchen. Z. Phys. 120, 513–538 (1943). [MF 8489]

The author develops the principles of a new theory intended to eliminate the difficulties in the present theory of elementary particles, such as infinite self-energies, infinite polarizability of the vacuum, etc. He investigates whether a quantity is actually "observable" and therefore fundamental in any future theory. On the other hand, he finds those quantities which lead to formal difficulties without necessarily being a link between the more directly observable quantities. For instance, the existence of a "smallest length" renders problematic all statements concerning more precise location. Actually statements about effective cross sections furnish only an a posteriori interpretation of experiments of scattering of particles. This suggests considering quantities, the definition of which is not affected by the existence of a smallest length, such as energies and momenta of particles, or the probability with which a particle of a given species might be produced or its energy and momentum might be changed.

On the basis of such considerations the author tries to establish a system of relations between the following quantities considered as in principle "observable": (1) the energy values of the stationary states of closed systems; (2) for the case of stationary collision, emission and absorption processes, the asymptotic behavior of the wave functions at great distances from the collision center. The mathematical formalism proposed conserves the structure of quantum-mechanical description. The traditional theory was based on a Hamiltonian with particle interaction potentials of a simple form. Now everything "observable" in a collision process can be expressed by a unitary matrix S. The square of the absolute value of the matrix element of S which belongs to the transition, say, from state A to state B determines the probability of finding the state B in the outgoing wave if the state A had been realized in the incident wave. The eigenvalues of the Hermitian matrix η correlated with S by $S=e^{i\eta}$ give the phase differences between the incident and the outgoing waves. If the interaction energy between the particles can be considered as a small perturbation, the matrix η is essentially identical with the submatrix of the interaction energy matrix which belongs to transitions between states of equal total energy and equal total momentum. However, even in the case of a Hamiltonian of simple form the matrix η is of a very complicated form and, moreover, its calculation leads to well-known divergences. The author argues that the assumption of a simple form for the Hamiltonian can hardly be justified, and he proposes rather to assume that the matrix η is of a simple form.

The present paper is confined to an investigation, in all generality, of the properties of the matrix η and its singularities. The new starting point has the following advantages. (1) The divergences can be entirely avoided. This is due to the fact that η contains only transition elements between states of the same energy. (2) The requirements of relativity can be fulfilled, whereas formerly the necessity of using some "cutting off" procedures has often led to difficulties. (3) The proposed scheme allows a rigorous discussion of rather complicated processes, such as the multiple processes of the shower phenomena, etc.

To the objection that, contrary to the Hamiltonian, the η-matrix is in general not known the author replies that the application of the correspondence principle presumes that the interaction can be treated as a small perturbation. In this case, also the matrix η is given by the correspondence argument. Therefore, on the basis of the correspondence principle, one may postulate with equal right a simple Hamiltonian or a simple η-matrix.

The investigation of the conditions is considered to be the central problem of the future theory of elementary particles. It seems possible that in this theory one should not speak of a Hamiltonian nor of a Schrödinger function of the usual form but only of something like the matrix η.

F. W. London (Durham, N. C.).

Heisenberg, W. Die beobachtbaren Grössen in der Theorie der Elementarteilchen. II. Z. Phys. 120, 673–702 (1943). [MF 8491]

In pursuance of the former general discussion [cf. the preceding review], the author investigates the physical implications of the assumption that instead of the Hamiltonian the η-matrix is given. Explicit calculations are made for the scattering of particles on the basis of different assumptions as to the interaction matrix. (1) A kind of δ-function interaction:

$$\eta=\text{const.} \int \varphi^4 dxdydzdt,$$

where $\varphi(x, y, z, t)$ is a scalar wave function fulfilling the commutation rules of Klein and Jordan [Z. Phys. 45, 751–765 (1927)]. The results conform in general with those of the usual theory. (2) Another form of interaction, leading to a finite cross-section in the limit of large energies. (3) The production of new particles is investigated on the basis of an interaction of the form

$$\eta=\text{const.} \int \varphi^6 dxdydzdt.$$

For small energies the results are again similar to those of the former theories. For high energies, however, multiple processes (showers) are encountered. Since no divergences appear the results obey rigorously the requirements of relativity. *F. W. London* (Durham, N. C.).

Balandin, A. A. Differential equation for the kinetics of contact monomolecular reactions investigated by the flow method. Acta Physicochim. URSS 17, 218–223 (1942). [MF 8403]

The author derives a linear first order differential equation for a catalytic reaction based on Langmuir's conception of the existence of a monomolecular boundary layer in which the reaction takes place. *I. Opatowski* (Chicago, Ill.).

DECEMBER ISSUE IS AN INDEX WHICH HAS
BEEN PHOTOGRAPHED AT THE BEGINNING
OF THE VOLUME(S).